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A Model of the Term Structure of Interest Rates with Fiscal-Monetary Policy Interactions

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Abstract

We study the implications of fiscal factors for the term structure of interest rates. We embed the flow budget constraint of the government into a general-equilibrium model of the bond yields. In our framework, the interaction between monetary and fiscal policy affects the ability of the government to meet the the solvency requirement. We assume that the tax rate is set according to a simple rule whereby taxes react proportionally to the outstanding liabilities of the government. A weak response of the fiscal authority to changes in public debt contributes to determine the inflation rate, thus acting as a driver of the term structure of interest rates. We depart from a discrete-time model that allows a clear-cut intuition, and price the term structure through the continuous-time limit. Since the model does not allow a closed-form solution, we use numerical methods to compute the prices of real and nominal zero-coupon bonds.

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> *"The default fear is kind of silly, however. The Greece government doesn't own a printing press, and it could default if it can't pay its bills. In sharp contrast, default risk is really the same thing as inflation risk in the U.S. "The U.S. can always pay its debts," says Harvey. "We can print money." That's not necessarily a good thing, of course, as the so-called monetization of the U.S. debt would debase the currency and destroy wealth."*
>
> Farrell [10]

1 Introduction

The recent events in financial markets have been characterized by a reassessment of risk for government bonds especially in Europe and the U.S. Several informal accounts suggest that this reaction by market participants may not only be due to enhanced default risk, but also to a risk of monetization of public debt by the central banks at a future point in time.

Standard monetary theories suggest a long-run link between the acceleration of money supply and the inflation rate. However, inflation risk may not be determined by central banks' actions, but also by the willingness of the fiscal authorities to use fiscal policy instrument to generate an appropriate level of primary surpluses. In other words, the interaction between monetary and fiscal policy altogether affects inflation expectations.

In this paper, we propose a general-equilibrium model of the term structure of interest rates where this type of interaction plays a central role for bond prices. Our aim is to study a modelling framework for the term structure where changes in fiscal fundamentals affect the solvency of the public sector. The evolution of these forces is then factored into the asset pricing decisions of rational agents.

We stress a recent interpretation of the determinants of fiscal solvency. The theory of price level determination advocated by Leeper [12], Sims [15], Woodford [17] and Cochrane [6] focuses on the role of interactions between fiscal and monetary policy in the determination of the inflation rate. In a nutshell, the idea is that the price level is determined by the degree of solvency of the government. If the expected primary surplus is not sufficient to comply with the intertemporal budget constraint of the government, then part of the public debt should be inflated away if it is default-free at some point in time. As a result, agents incorporate this source of inflation risk into bond prices. Summing up, under this interpretation, the interaction between monetary and fiscal policy affect government solvency and, in turn, inflation expectations, thus acting as a driver of the term structure of interest rates.

Although the fiscal theory has generated a substantial debate on the capability of fiscal and monetary policy to affect the price level, only a few studies have considered its implications for asset prices. This considerations holds both for the finance and macroeconomics literature. For instance, the continuous-time model of the term structure of interest proposed by Buraschi [4] and Buraschi and Jiltsov [5] includes lump-sum taxes, but disregards the implications of the government budget constraint. Dai and Philippon [9] estimates a no-arbitrage affine term structure model with fiscal variables on US data. They find significant responses of the term structure of interest rates to the deficit-GDP ratio. The macroeconomic restrictions they impose to identify the structural responses are fairly different from those implied by the fiscal theory of the price level (e.g., see Sala [14]).

The available finance models the term structure of interest rates consider an explicit role for only two crucial factors, output growth and monetary policy, which is typically expressed as a diffusion process for the growth of money supply. In this paper, we consider a general-equilibrium model with money where the flow budget constraint of the government plays an active role. This provides a link between monetary and fiscal policy because lump-sum taxes are adjusted as a function of real debt. We solve the structural model, and derive the law of motion for the nominal and real interest rates. We also study how the term structure responds to the fiscal parameters.

This paper is organized as follows. The second section provides a short account of the determination of inflation according to the fiscal theory. The third section discusses the model framework. Section 4 presents the equilibrium conditions. In section 5, we consider a specialized economy with a realistic set of functional assumptions that allow a solution. In section 6 we present the pricing of the term structure of government bonds. Since the solution does not admit a closed form, we use numerical simulations in section 7 to generate some qualitative results. Section 8 reports some concluding remarks.

2 The determination of inflation: the approach of the fiscal theory

In the standard monetary theory, the price level affects both the demand for money, and aggregate supply. The equilibrium level of money according to the equation

$$MV = PY \tag{1}$$

Equation (1) is the traditional quantity theory of money and V indicates money velocity (usually expressed as a function of the nominal interest rate). Coupled with (1), standard models also include a government budget constraint

$$\frac{M_{t-1} + B_{t-1}}{P_t} = E_t \sum_{j=0}^{\infty} \beta^j s_{t+j} \tag{2}$$

where s_{t+s} indicates the expected future surpluses, while M_{t-1} B_{t-1} indicate money supply and government debt, respectively. Equation (2) says that the sum of nominal debt inclusive of money supply must be equal to the present value of future primary surpluses (net of interest rates).

The key issue is to understand if equation (30) can be considered as an equilibrium relationship or a constraint. Traditionally, the government, given the level of prices P_t must adjust the future surpluses in a way that equation (2) has to hold. The fiscal theory challenges this conventional wisdom about price level determination.

Apart from Smith (1776), an interesting analysis of the current approach has been considered by Lerner (1947): "...if the state is willing to accept the proposed money in payment of taxes and other obligations to itself, the trick is done. Everyone who has obligations to the state will be willing to accept the pieces of paper with which he can settle the obligations, and all the other people will be willing to accept these pieces of paper because the know that the taxpayers, etc., will be willing to accept them in turn."

The basic logic of the monetary theory is related to the commitment of the government for the redemption of fiat money. In the context of the fiscal theory, this refers to the ability of the government to commit to peg the primary surplus in order to preserve the value of the real debt constraint (from equation (2)).

The key element of fiscal theory is the different emphasis attached to the government behavior. In fact, according to this approach, the government is forced to adjust the prospective surpluses in order to respect the equilibrium relationship given in (2). To properly understand this aspect, let us recall the analogy with the stock evaluation equation: if market pushes stock's price of company Y up, this does not necessarily mean that company Y is forced to raise the prospective earnings or dividends. Any kind of reaction of company Y to its own stock price movements does not have to be thought as the result of interpreting relation (2) as a budget constraint. In the same fashion, if a sudden increase of the price level reduces the real value of the outstanding debt, this does not necessarily forces the government to cut the prospective surpluses. All what matters is that the government keeps the commitment to repay nominal debt and that the official currency is still accepted for tax payments. The key point here is that whatsoever kind of reaction the government might enact, that reaction is never forced by the logic of a budget constraint.

3 The model

We study an economy populated by a representative agent that maximizes over the composition of her portfolio along the lines of the traditional literature on consumption and asset pricing. We model the economy at discrete time intervals of length Δt. The representative agent chooses its portfolio holdings by maximizing the following utility function

$$\sum_{t=0}^{\infty} e^{-\beta t} E_0 \left\{ u \left(C_t, \frac{M_t}{P_t} \right) \right\} \Delta t \tag{3}$$

where β is the discount factor. In equation (3), C_t indicates the level of consumption over the interval $[t, t+\Delta t]$, M_t is the nominal money stock providing utility to the representative agent over the interval of length $[t-\Delta, t]$, and P_t is the price of the consumption good. Real money balances M_t/P_t enter the utility function of the household. The utility function is twice continuously differentiable and concave in both consumption and real balances: $u_c > 0$ and $u_m > 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cm} < 0$ and $u_{cc}u_{mm} - (u_{cm})^2 > 0$, where the subscript to u indicates the partial derivative. We make the following functional assumption on the utility function

$$u\left(C_t, \frac{M_t}{P_t}\right) = \phi \log C_t + (1-\phi)\log\left(\frac{M_t}{P_t}\right) \tag{4}$$

This type of utility function is used in Stulz [16]. In equation (4), the preference parameter ϕ is chosen so that the nominal and real spot rates determined under the assumption of absence of arbitrage opportunities are also equilibrium values (see Corollary 1 in the Appendix).

As a working hypothesis to derive the first order conditions, we consider a model of pure endowment economy where output growth evolves as

$$\frac{\Delta Y_t}{Y_t} = \frac{Y_{t+\Delta t} - Y_t}{Y_t} = \mu_{Y,t}\Delta t + \sigma_{Y,t}\Omega_{Y,t}\sqrt{\Delta t}. \tag{5}$$

The terms $\mu_{Y,t}$ and $\sigma_{Y,t}$ are, respectively, the conditional expected value and the standard deviation of output per unit of time and $\{\Omega_{Y,t} t = 0, \Delta t, \ldots\}$ is a standard Normal process.

3.1 Fiscal and monetary policy

The main point of this paper is to examine the impact of the interaction between monetary and fiscal policy on the the term structure of interest rates. We think of 'interactions' in the sense captured by the "fiscal theory of the price level" of Leeper [12], Sims [15], Woodford [17], and recently extended by Cochrane [6, 7], which suggests that a tight fiscal policy is a necessary complement to ensure price stability.

We define the money supply aggregate (in nominal terms) as

$$M_t^s = H_t + F_t. \tag{6}$$

In equation (6) we observe that the total money supply is determined by two components. H_t is the so called 'high powered money' (or monetary base).

F_t represents the amount of money needed by the government to budget its balance. Basically, F_t is an additional financing source for the government apart from taxes and debt.

We assume that H_t and F_t follow the processes described by

$$\frac{\Delta H_t}{H_t} = \frac{H_{t+\Delta t} - H_t}{H_t} = \mu_{H,t}\Delta t \tag{7}$$

$$\frac{\Delta F_t}{F_t} = \frac{F_{t+\Delta t} - F_t}{F_t} = \mu_{F,t}\Delta t + \sigma_{F,t}\Omega_{F,t}\sqrt{\Delta t} \tag{8}$$

where $\mu_{H,t}$ and $\mu_{F,t}$ are, respectively, the mean of the stochastic process of the monetary base and of the financing to public debt. In (7), the stochastic process for H_t does not have a standard error term, implying that the monetary base possesses only a deterministic component. The process leading F_t, instead, has a standard deviation term $\sigma_{F,t}$, where $\{\Omega_{F,t} t = 0, \Delta t, \ldots\}$ are standard Normal random variables.

From (6), (7) and (8), we can write the stochastic process for the total money supply M^s

$$\frac{\Delta M_t^s}{M_t^s} = \frac{M_{t+\Delta t}^s - M_t^s}{M_t^s} = \mu_{M,t}\Delta t + \sigma_{M,t}\Omega_{M,t}\sqrt{\Delta t} \tag{9}$$

$$\mu_{M,t} = \mu_{H,t} + \mu_{F,t} \tag{10}$$

$$\sigma_{M,t}\Omega_{M,t} = \sigma_{F,t}\Omega_{F,t}. \tag{11}$$

At a first glance, these expressions stress that the central bank is assumed to target money growth.

The subsequent building block of the model assigns a proper macroeconomic role to the government. The innovation introduced in this paper with respect to the existing literature consists in the key role for the government budget constraint

$$\Delta D_{t+\Delta t} + \Delta F_{t+\Delta t} = \Delta i_{t+\Delta t} D_t - \Delta T_{t+\Delta t} \tag{12}$$

where D_t indicates the stock of public debt, and $\Delta i_{t+\Delta t}$ is the stochastic process of the nominal spot interest rate, whose endogenous law of motion will be computed later. Moreover, $\Delta T_{t+\Delta t}$ is the stochastic process for taxes. We assume that the government does not face any form of public spending. Recall that $\Delta D_{t+\Delta t} = D_{t+\Delta t} - D_t$, $\Delta F_{t+\Delta t} = F_{t+\Delta t} - F_t$. Basically, the government can use taxes, money and debt to finance its budget.

Following the fiscal theory of price level, we assume that the government sets taxes according to the simple rule rule

$$\Delta T_{t+\Delta t} = \phi_1 D_t \Delta t + \phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t} \tag{13}$$

According to (13), the government sets as a function of the outstanding amount of public debt. This means that if the stock of debt issued rises, taxes must change accordingly with a marginal elasticity equal to ϕ_1. A bound on ϕ_1 can be established from Sims [15] by setting ϕ_1 at a value lower than or equal to the discount factor β.

To close the model, we assume that the process for the nominal spot interest rate is

$$\Delta i_{t+\Delta t} = \mu_i \Delta t + \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t}, \tag{14}$$

for values of the mean and the standard deviations to be determined later. By plugging (14) and (13) into (12), we can recover the flow budget constraint of the public sector

$$\Delta D_{t+\Delta t} + \Delta F_{t+\Delta t} = (\mu_i - \phi_1) D_t \Delta t + D_t \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t} - \phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t}. \tag{15}$$

In order to obtain a semi-closed form solution, we assume that the quantity of newly-issued public debt follows a deterministic process with mean μ_D

$$\frac{\Delta D_t}{D_t} = \frac{D_{t+\Delta t} - D_t}{D_t} = \mu_{D,t} \Delta t. \tag{16}$$

Thus, the flow budget constraint becomes

$$\begin{aligned} \mu_{D,t} \Delta t + \mu_{F,t} \Delta t + \sigma_{F,t} \Omega_{F,t} \sqrt{\Delta t} = {} & (\mu_i - \phi_1) D_t \Delta t \\ & + D_t \sigma_{i,t} \Omega_{i,t} \sqrt{\Delta t} + -\phi_1 D_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t}. \end{aligned} \tag{17}$$

To get intuition on these relations, we focus on their deterministic part. Assume that the government aims to maintain a constant ratio of nominal bond to money, i.e., $\psi = D/F$. Therefore, by applying Ito's Lemma to the definition of ψ, we can write the relationship between the mean of the public debt and money

$$\mu_D = \mu_F - \sigma_F^2. \tag{18}$$

From the equality between the deterministic and the stochastic terms of F_t,

$$\psi \mu_D + \mu_F = (\mu_i - \phi_1) \psi \tag{19}$$

$$\sigma_F = \psi (\sigma_i - \phi_1 \sigma_T). \tag{20}$$

Finally, using the definition of μ_D into (19), we obtain the semi-closed solution for the mean of the stochastic process for money

$$\mu_F = \frac{\left(\mu_i - \phi_1 + \sigma_F^2\right)\psi}{1+\psi}. \tag{21}$$

Therefore, (20) and (21) represent the full equilibrium relationship in the economy. By using (21), it is clear that the mean of the stochastic process leading money is

$$\mu_M = \mu_H + \frac{\left(\mu_i - \phi_1 + \sigma_F^2\right)\psi}{1+\psi}. \tag{22}$$

3.2 The optimal choice problem

The representative agent's budget constraint is

$$\begin{aligned} M_t + \left(P_{z,t} + P_t^C y_t \Delta t\right) z_t + P_t^C a_{1,t} + a_{2,t} + \sum_{i=3}^{N} P_{i,t} a_{i,t} &= P_t^C C_t \Delta t + M_{t+\Delta t} \\ &+ P_{z,t} z_{t+\Delta t} + P_t^C \frac{a_{1,t+\Delta t}}{1+r_t \Delta t} + \frac{a_{2,t+\Delta t}}{1+i_t \Delta t} + \sum_{i=3}^{N} P_{i,t} a_{i,t+\Delta t} \end{aligned} \tag{23}$$

The investor can choose among one real and one nominal bond (both risk free), and $N-2$ equities. Each bond is issued at time t and has maturity at time $t+\Delta t$. The return on bond are i_t for the nominal bond, and r_t for the real bond. $P_{i,t}$ is the price (inclusive of dividends) of asset i at time t. The representative agent demands M_t, for cash, C_t for consumption and x_t for equity holdings. $a_{1,t}, a_{2,t}, \ldots a_{N,t}$ represent the unit of financial asset held from $(t-\Delta t)$ to t.[4]

The choice problem of the representative investor consists in the maximization of the utility function (4) subject to the budget constraint (23). The first order conditions for C_t, $a_{1,t}$, $a_{2,t}$, M_t and $a_{i,t}$ are, respectively,

$$u_c\left(C_t, m_t\right) = \lambda_t P_t^C \tag{24}$$

$$E_t\left[e^{-\beta \Delta t} \lambda_{t+\Delta t} P_{t+\Delta t}^C \left(1 + r_t \Delta t\right)\right] = \lambda_t P_t^C \tag{25}$$

[4] This setup above described is similar to that of Baksi and Chen [1], who use this model to study the impact of monetary policy and inflation on financial asset.

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}\left(1+i_t\Delta t\right)\right]=\lambda_t \tag{26}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}+u_m\left(C_{t+\Delta t},m_{t+\Delta t}\right)\frac{1}{P^C_{t+\Delta t}}\right]=\lambda_t \tag{27}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}P_{i,t+\Delta t}\right]=\lambda_t P_{i,t} \tag{28}$$

4 Definition of equilibrium

We assume that the economy is populated by identical agents. In a representative agent economy, optimal consumption, money demand and portfolio holdings must adjust in order that the following equilibrium conditions are verified in general equilibrium

$$C_t=Y_t \tag{29}$$

$$M_t\equiv M^s_t=M^d_t \tag{30}$$

with $z_t = 1$ and $a_{i,t} = 0 \quad \forall\, i = 1,\dots N$. In a pure endowment economy, (29) states that the total amount of consumption must equal the total output endowment. The equality between money demand and supply is stated in equation (30). Moreover, each agent's demand for equity shares must equal the supply.

Using equations (29)–(30) and the first order conditions (24)–(28), we obtain

$$u_c\left(Y_t,m_t\right)=e^{-\beta\Delta t}E_t\left[u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)\left(1+r_t\Delta t\right)\right] \tag{31}$$

$$\frac{u_c\left(Y_t,m_t\right)}{P^C_t}=e^{-\beta\Delta t}E_t\left[\frac{u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)}{P^C_{t+\Delta t}}\left(1+i_t\Delta t\right)\right] \tag{32}$$

$$u_c\left(Y_t,m_t\right)=e^{-\beta\Delta t}E_t\left[u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)\frac{p_{i,t+\Delta t}}{p_{i,t}}\right] \tag{33}$$

$$u_c\left(Y_t,m_t\right)=e^{-\beta\Delta t}E_t\left\{\left[u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)+u_m\left(Y_{t+\Delta t},m_{t+\Delta t}\right)\right]\frac{P^C_t}{P^C_{t+\Delta t}}\right\} \tag{34}$$

where $m_t\equiv\frac{M_t}{P^C_t}$ is the real cash balances and $p_{i,t}\equiv\frac{P^i_t}{P^C_t}$ is the real price (in terms of the consumption goods) of asset i at time t.

Equations (31)–(34) are Euler conditions derived from the utility maximization problem of the representative investor. Equations (31) and (34) state that the representative investor must be indifferent between investing an amount of money equal to P_t^C in a real risk-free bond and holding the same amount in cash. This arbitrage condition holds also for nominal bonds (see equations (32) and (34)). Finally, equations (33) and (34) describe the relation of indifference between investing one more amount of cash of size P_t^C in asset i and holding the same amount in a pure cash. Equations (31)–(34) establish the demand for real money, while (32) and (34) yield the demand for money in nominal terms. Equation (34) states that, in equilibrium, the agent is indifferent between holding P_t^C amount of cash and consuming one extra unit of the good, because both actions produce the same marginal utility. The link between the price level and monetary policy is established by equation (34). Monetary policy and the asset market are tied together through equations (34) and (33), which establish the consistency between the money supply and asset markets. Finally, the interdependence between monetary policy and the goods market is described by equations (31) and (34).

Equation (33) must to hold also for the equity Z_t when we replace $\frac{p_{i,t+\Delta t}}{p_{i,t}}$ with $\frac{p_{z,t+\Delta t}+Y_{t+\Delta t}}{p_{z,t}}$, where $p_{z,t} = \frac{P_{z,t}}{P_t^C}$ is the real price of the equity share. Additional sufficient conditions for the existence of an interior optimum are the two transversality conditions

$$\lim_{T \longrightarrow \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c\left(Y_T, m_T\right)}{u_c\left(Y_t, m_t\right)} p_{i,t} \right\} = 0 \tag{35}$$

$$\lim_{T \longrightarrow \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c\left(Y_T, m_T\right)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_t^C} \right\} = 0 \tag{36}$$

The equality (35) rules out bubbles in the price level of any risky asset. Condition (36), instead, prevents bubbles in the price level from taking place. The intuition behind the two TVCs is if (35) is violated the agent is willing to sacrifice actual consumption in favor of future consumption derived from proceeds from investment in risky assets without bound. Under condition (36), the agent accepts a reduction in consumption today in exchange for a larger amount of money in the future without bound.

To conclude the characterization of the equilibrium relations, we need to define the stochastic process for real asset prices. We follow Baksi and Chen [1], Merton [13] and Grossman and Shiller [11] by assuming

$$\frac{\Delta p_{i,t}}{p_{i,t}} = \mu_{i,t}^e \Delta t + \sigma_{i,t}^e \Omega_{i,t}^e \sqrt{\Delta t} \tag{37}$$

where $\mu^e_{i,t}$ and $\sigma^e_{i,t}$ are, respectively, the conditional expected value and the standard deviation of real return on asset i per unit of time. Finally, the process $\left\{\Omega^e_{i,t} t = 0, \Delta t, \ldots\right\}$ is a standard Normal.

4.1 The equilibrium in the continuous time limit

In this section we characterize the equilibrium for the continuous time limit. These results are independent from the assumptions made on the role of fiscal and monetary policies in the determination of the equilibrium. For this reason, the results presented here are similar to those discussed in Baksi and Chen [1] and Balduzzi [2].

Proposition 1. *The equilibrium risk premiums for any risky asset over the real spot interest rate is*

$$\mu^e_{i,t} - r_t = -\frac{C_t u_{cc}}{u_c} cov_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dY_t}{Y_t}\right) - \frac{m_t u_{cm}}{u_c} cov_t\left(\frac{dp_{i,t}}{p_{i,t}}, \frac{dm_t}{m_t}\right). \tag{38}$$

Proof. Subtract equation (31) from (33), manipulating the resulting expression and using the definition of the stochastic process for $p_{i,t}$ (37), we obtain

$$e^{-\beta\Delta t} E_t \left\{\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)}\left[\left(\mu^e_{i,t} - r_t\right)\Delta t + \sigma^e_{i,t}\Omega^e_{i,t}\sqrt{\Delta t}\right]\right\} = 0. \tag{39}$$

Thus, by taking a Taylor expansion of the equation (39) around steady state, we have

$$e^{-\beta\Delta t} E_t \left\{\left[\left(\mu^e_{i,t} - r_t\right)\Delta t + \sigma^e_{i,t}\Omega^e_{i,t}\sqrt{\Delta t}\right]\right. \tag{40}$$

$$\left.\times\left[1 + \frac{u_{cc}\left(Y_{t+\Delta t}, m_{t+\Delta t}\right) Y_t}{u_c\left(Y_t, m_t\right)}\frac{\Delta Y_t}{Y_t} + \frac{u_{cm}\left(Y_{t+\Delta t}, m_{t+\Delta t}\right) m_t}{u_c\left(Y_t, m_t\right)}\frac{\Delta m_t}{m_t}\right]\right\}\frac{1}{\Delta t} = 0. \tag{41}$$

By letting $\Delta t \to 0$ and applying Ito's multiplication rule, we obtain (38). □

Proposition 2. *The real price for the equity share is*

$$P_{z,t} = E_t \int_t^\infty e^{-\beta(s-t)}\frac{u_c\left(Y_s, m_s\right)}{u_c\left(Y_t, m_t\right)} Y_s ds \tag{42}$$

Proof. Recalling that

$$\frac{P_{z,t+\Delta t} + Y_{t+\Delta t}\Delta t}{P_{z,t}} = \frac{\Delta p_{i,t+\Delta t}}{p_{i,t}}, \tag{43}$$

use equation (33) to obtain

$$P_{z,t} = E_t \left\{ e^{-\beta\Delta t} \frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \left(Y_{t+\Delta t}\Delta t + P_{x,t+\Delta t}\right) \right\}. \tag{44}$$

After iterating forward, the result is

$$P_{z,t} = E_t \sum_{j=1}^{\infty} e^{-\beta(j\Delta t)} \frac{u_c\left(Y_{t+j\Delta t}, m_{t+j\Delta t}\right)}{u_c\left(Y_t, m_t\right)} Y_{t+j\Delta t}\Delta t. \tag{45}$$

Thus, by taking the limit for $\Delta t \to 0$ in (45) we finally get the result under (42). □

Proposition 3. *In the continuous time limit equilibrium, the commodity price level is given at time t by*

$$\frac{1}{P_t^C} = E_t \int_t^{\infty} e^{-\beta(s-t)} \frac{u_m(Y_s, m_s)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_s^C} ds. \tag{46}$$

The expected inflation rate is

$$\begin{aligned} \pi_t &\equiv \frac{1}{dt} E_t \left\{ \frac{dP_t^C}{P_t^C} \right\} \\ &= i_t - r_t + var_t \left\{ \frac{dP_t^C}{P_t^C} \right\} - \frac{u_{cc} Y_t}{u_c} cov_t \left(\frac{dY_t}{Y_t}, \frac{dP_t^C}{P_t^C} \right) \\ &\quad - \frac{u_{cm} m_t}{u_c} cov_t \left(\frac{dP_t^C}{P_t^C}, \frac{dm_t}{m_t} \right). \end{aligned} \tag{47}$$

Proof. Rewrite the first order condition (34) as follows

$$\frac{1}{P_t^C} = e^{-\beta\Delta t} E_t \left\{ \left[\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \frac{1}{P_{t+\Delta t}^C} + \frac{u_m\left(C_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \right] \frac{\Delta t}{P_{t+\Delta t}^C} \right\} \tag{48}$$

then, iterate equation (48) to get

$$\frac{1}{P_t^C} = E_t \left\{ \sum_{j=1}^{\infty} e^{-\beta(j\Delta t)} \frac{u_m\left(C_{t+j\Delta t}, m_{t+j\Delta t}\right)}{u_c\left(Y_t, m_t\right)} \frac{\Delta t}{P_{t+j\Delta t}^C} \right\}. \tag{49}$$

Taking the limit of the equation (49) we get the result under (46).

To compute the inflation rate, divide the first order conditions (31) and (32)

$$E_t\left\{\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)}\left(1+r_t\Delta t\right)\right\} = E_t\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\left(1+i_t\Delta t\right)\frac{P_t^C}{P_{t+\Delta t}^C}\right]. \tag{50}$$

After taking the Taylor approximation of (50) and re-arranging,

$$\begin{aligned}\left(i_t-r_t\right)\Delta t = &\left[1+\frac{u_{cc}Y_t}{u_c}\left(\frac{\Delta Y_t}{Y_t}\right)+\frac{u_{cm}}{u_c}m_t\left(\frac{\Delta m_t}{m_t}\right)\right]\\ &\times\left[\frac{\Delta P_t^C}{P_t^C}-\left(\frac{\Delta P_t^C}{P_t^C}\right)^2\right]+o\left(\Delta t\right)^{3/2}. \end{aligned}\tag{51}$$

We take the limit of equation (51) for $\Delta t \to 0$ and obtain

$$\begin{aligned} i_t - r_t = &\frac{1}{dt}E_t\left\{\frac{dP_t^C}{P_t^C}\right\} - var_t\left\{\frac{dP_t^C}{P_t^C}\right\}+\frac{u_{cc}Y_t}{u_c}cov_t\left(\frac{dY_t}{Y_t},\frac{dP_t^C}{P_t^C}\right)\\ &+\frac{u_{cm}m_t}{u_c}cov_t\left(\frac{dP_t^C}{P_t^C},\frac{dm_t}{m_t}\right). \end{aligned}\tag{52}$$

Finally, by using $\pi_t = \frac{1}{dt}E_t\left\{\frac{dP_t^C}{P_t^C}\right\}$ and rearranging, we obtain equation (47). □

5 A specific model economy

In what follows we lay out the assumptions used to derive the stochastic processes for the price level and the other variables. We assume that the evolution of output follows from

$$dY_t = \left(\mu_Y + \eta_Y x_t\right)dt + \sigma_Y\sqrt{x_t}dW_{x,t}. \tag{53}$$

where the process for the technology factor x_t is

$$dx_t = a\left(b - x_t\right)dt + \sigma_x\sqrt{x_t}dW_{x,t}, \tag{54}$$

where $\left(W_{x,t}\right)_t$ is a unidimensional $\mathbb{Q}$-Brownian motion, μ_Y, η_Y, σ_Y, a, b, and σ_x are fixed real numbers.

The monetary aggregates H_t and F_t follow the exogenous processes

$$d\ln H_t = \mu_H^* dt + d\ln(q_t) \tag{55}$$

$$d\ln F_t = \overline{\mu}_F dt + d\ln(q_t) \tag{56}$$

where q_t is the detrended money supply process. Each type of money supply has two components, a drift term and a stochastic part. In particular, μ_H^* is assumed to be constant and positive, while $\overline{\mu}_F$ is determined by equation (21). From Baksi and Chen [1], q_t evolves according to

$$\frac{dq_t}{q_t} = k_q(\mu_q - q_t)\,dt + \sigma_q\sqrt{q_t}dW_{i,t}, \quad i = H, F \tag{57}$$

where $(W_{i,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent upon $(W_{x,t})_t$. Therefore, by using the definition of money supply (6), we find that the stochastic process leading money supply is

$$\frac{dM_t}{M_t} = \mu_{M,t}dt + \sigma_q\sqrt{q_t}dW_{M,t}. \tag{58}$$

$(W_{M,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent from $(W_{x,t})_t$ and $(W_{i,t})_t$, and where

$$\mu_M = \mu_M^* + 2k_q(\mu_q - q_t), \tag{59}$$

with $\mu_M^* = \mu_H^* + \overline{\mu}_F$, and $d\Omega_{M,t} = d\Omega_{H,t} + d\Omega_{F,t}$. These assumptions allow us to compute the equilibrium price level of the commodity and the inflation process.

Theorem 4. *Given the utility function of the representative agent as described by equation (4), then the equilibrium price level is*

$$P_t^c = \frac{\phi}{1-\phi}\frac{q_t^2(\beta+\mu_M^*)(\beta+\mu_M^*+2k_q\mu_q)}{(\beta+\mu_M^*)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}. \tag{60}$$

The stochastic process of the consumer price index is

$$\frac{dP_t^c}{P_t^c} = \pi_t dt + \sigma_q\sqrt{q_t}\left[1 + \frac{(\Delta_q\Psi - \Delta\Psi_q)}{\Delta\Psi}q_t\right]dW_{M,t} - \sigma_y\sqrt{x_t}dW_{x,t} \tag{61}$$

where the inflation rate is

$$\begin{aligned}\pi_t &= \mu_M^* - \mu_y + \left(\sigma_y^2 - \eta_y\right) x_t \\ &+ \frac{\left(\Delta_q \Psi - \Delta \Psi_q\right)}{\Delta \Psi} q_t \left(k_q \left(\mu_q - q_t\right) + \frac{\sigma_q^2 q_t}{2}\right) \\ &+ \frac{\left[2\left(\Delta_{qq}\Psi - \Delta\Psi_{qq}\right) - \Delta_q \Psi + \Delta \Psi_q\right] \sigma_q^2 q_t^{3/2}}{2\Psi^2 \Delta} \qquad (62)\\ \Delta(q) &= \frac{\phi}{1-\phi}\left[q_t^2 \left(\beta + \mu_M^*\right)\left(\beta + \mu_M^* + 2k_q\mu_q\right)\right] \qquad (63)\\ \Psi(q) &= \left(\beta + \mu_M^*\right) + \left(k_q + 3\sigma_q^2\right) q_t 2k_q\mu_q \qquad (64)\end{aligned}$$

and $\Delta_q = \frac{\partial \Delta(q)}{\partial q}$, $\Psi_q = \frac{\partial \Psi(q)}{\partial q}$, *and* $\Delta_{qq} = \frac{\partial \Delta(q)}{\partial q}$.

Proof. We start by showing how to get (60). From (58) we have that

$$\frac{1}{M_t} = e^{-\mu_M^*} q_t^{-2}. \qquad (65)$$

Define $G(q) = \frac{1}{q_t^2}$. Thus by using Ito's Lemma, we obtain

$$d\left[\frac{e^{2k_q\mu_q}}{q_t^2}\right] = \frac{2}{q_t}\left(k_q + 3\sigma_q^2\right) e^{2k_q\mu_q t} dt - \frac{2\sigma_q}{q_t\sqrt{q_t}} dW_{M,t}. \qquad (66)$$

The expected value is

$$\begin{aligned}E_t\left[\frac{1}{q_s^2}\right] &= E_t\left[\frac{1}{q_s^2} \mid q_t\right] = \qquad (67)\\ &= E_t\left[e^{-2k_q\mu_q s}\left\{\int_t^s d\left[\frac{e^{2k_q\mu_q z}}{q_z^2}\right] + \frac{e^{2k_q\mu_q t}}{q_t^2}\right\} \mid q_t\right] = \qquad (68)\\ &= \frac{e^{-2k_q\mu_q(s-t)}}{q_t} + \frac{\left(k_q + 3\sigma_q^2\right)}{q_t k_q \mu_q}\left(1 - e^{-2k_q\mu_q(s-t)}\right). \qquad (69)\end{aligned}$$

Finally, from the first order conditions of the problem of the representative agent, we get

$$\frac{1}{P_t^C Y_t} = \frac{1-\phi}{\phi}\int_t^\infty E_t\left[\frac{1}{q_s^2}\right] e^{-\beta(s-t)-\mu_M^*} ds. \qquad (70)$$

After solving for the integral, we find equation (61).

To compute the inflation rate, it is enough to apply Ito's lemma to (61) by setting $V\left(M_t, Y_t, q_t\right) = \frac{\Delta(q_t)}{\Psi(q_t)}\frac{M_t}{Y_t}$ so that

$$dP_t^c = G_M dM_t + G_Y dY_t + G_q dq_t + \frac{1}{2}\left[G_{YY}\left(dY_t\right)^2 + G_{qq}\left(dq_t\right)^2 + G_{Mq}\left(dM_t\right)\left(dq_t\right)\right]. \qquad (71)$$

By taking into account (63), (64), and the definitions for Y_t, q_t and M_t, (53), (57) and (58) respectively, we obtain equation (62). □

6 Pricing the term structure

6.1 The real spot interest rate

In this section we derive the dynamics of the real spot interest rate. Since this requires solving the differential equation (54), we assume that x_t is Markov and satisfies the necessary technical conditions to apply the Representation Theorem of Feyman-Kac.[5] As a result, we can use the partial differential equation — PDE — approach to compute the real spot rate.

Theorem 5. *If the technology process follows (54) , then the real spot rate* r_t *is a function* $\phi(t, x_t)$ *that represents the unique solution to the Kolmogorov PDE*

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2 \phi(t,x_t)}{\partial x_t^2} + a(b - x_t)\frac{\partial \phi(t,x_t)}{\partial x_t} + \frac{\partial \phi(t,x_t)}{\partial t} - x_t \phi(t, x_t) = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma + a} \end{cases} \tag{72}$$

where the final condition is the long time spot rate determined in Cox, Ingersoll and Ross [8]. This gives

$$r_t = A(\theta) e^{C(\theta) x_t} \tag{73}$$

where $\theta = T - t$ *and*

$$C(\theta) = \frac{\sigma_x^2 (2 + a)\left(1 - e^{\gamma\theta}\gamma\right)}{2a\left[\sigma_x^2 - a + e^{\gamma\theta}\left(a - \sigma_x^2\right) - \left(1 + e^{\gamma\theta}\right)\gamma\right]} \tag{74}$$

$$A(\theta) = \frac{1}{\gamma + a}\left\{2^{\chi} a b e^{\nu(\theta)} \gamma^{\zeta}\left[a + \gamma - \sigma_x^2 + e^{\gamma\theta}\left(\gamma - a + \sigma_x^2\right)\right]^{-\zeta}\right\} \tag{75}$$

$$\chi = \frac{a^2 - \gamma^2 - 2\sigma_x^2(a + b + ab) + \sigma^4(1 + b)}{\left(a - \sigma_x^2\right)^2 - \gamma^2} \tag{76}$$

$$\nu(\theta) = \frac{b\theta\sigma_x^2(2 + a - \gamma)}{2\left(a + \gamma - \sigma_x^2\right)} \tag{77}$$

$$\zeta = \frac{b\sigma_x^2\left(\sigma_x^2 - 2a - 2\right)}{\left(a - \sigma^2\right)^2 - \gamma^2} \tag{78}$$

$$\gamma = \sqrt{a^2 + 2\sigma_x^2} \tag{79}$$

Proof. If $\phi(t, x_t) = r_t$, like in equation (73), then the Kolmogorov PDE is

$$\begin{cases} \frac{1}{2}\sigma^2 x_t C^2(\theta) r_t + a(b - x_t) C(\theta) r_t + A'(\theta) e^{C(\theta) x_t} + \\ + C'(\theta) x_t r_t - x_t r_t = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma + a} \end{cases} \tag{80}$$

[5] The drift and volatility terms in (54) must be Lipschitz and bounded on $\Re$.

where $A^{'}(\theta)$ and $C^{'}(\theta)$ represent the derivative with respect to time of functions $A(\theta)$ and $C(\theta)$ equations (74) and (75). Given that the Kolmogorov PDE is verified for all t and x_t, we can divide it into two parts. One part is dependent and the other one is independent from x_t. The task boils down to solving the differential equation — DE — system

$$\begin{cases} \frac{1}{2}\sigma_x^2 C^2(\theta) - aC(\theta) + C^{'}(\theta) - 1 = 0 & C(0) = 1 \\ A^{'}(\theta) + abA(\theta)C(\theta) = 0 & A(0) = \frac{2ab}{\gamma+a}. \end{cases} \tag{81}$$

The first DE is a Riccati equation whose solution is (74). The second equation, instead, is a regular first order DE with solution given by (75). □

Lemma 6. *The dynamics of real spot interest rate is*

$$dr_t = a^*(\theta)\left[b^*(\theta) - r_t\right]dt + C(\theta)\, r_t \sigma_x \sqrt{x_t}\, dW_{x,t} \tag{82}$$

where $(W_{x,t})_t$ *is a unidimensional* $\mathbb{Q}-$*Brownian motion, and with*

$$a^*(\theta) = -\left[a(b - x_t)C(\theta) + C^{'}(\theta)x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right] \tag{83}$$

$$b^*(\theta) = \frac{A^{'}(\theta)e^{C(\theta)x_t}}{a^*(\theta)}. \tag{84}$$

Proof. By applying Ito's Lemma to (73), the dynamics of $r_t = \phi(t, x_t)$ is

$$\begin{aligned} dr_t &= C(\theta)r_t dx_t + \frac{\partial r_t}{\partial t}dt + \frac{1}{2}\sigma_x^2 x_t C^2(\theta) r_t dt \\ &= r_t\left[a(b - x_t)C(\theta) + C^{'}(\theta)x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right]dt + \\ &\quad +A^{'}(\theta)e^{C(\theta)x_t}dt + C(\theta)r_t\sigma_x\sqrt{x_t}\,dW_{t,x}. \end{aligned} \tag{85}$$

Finally, we can write (85) as a mean reverting process in square root with time dependent coefficients like in (82). □

6.2 The term structure of real interest rates

Here we show how to compute the price of zero coupon bonds as a function of time, technology and the real spot rate. We follow again a PDE approach because both x_t and r_t are Markov and satisfy the necessary technical conditions to apply the Representation Theorem of Feyman-Kac. The solution has no closed form, and it is necessary to use numerical methods to understand how it works.

Theorem 7. *If technology and the real spot rate follow (54) and (85), respectively, then the zero coupon bond $B(t,T)$ is a function $\vartheta(t,x_t,r_t)$ that represents the unique solution to the Kolmogorov PDE*

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial x_t^2} + a\left[b-x_t\right]\frac{\partial\vartheta(t,x_t,r_t)}{\partial x_t} + \frac{1}{2}C^2(\theta)\,C^{*2}(\theta)\,r_t^2\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial r_t^2} + \\ +a^*(\theta)\left[b^*(\theta)-r_t\right]\frac{\partial\vartheta(t,x_t,r_t)}{\partial r_t} + r_t C(\theta)\,\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,r_t)}{\partial r_t \partial x_t} + \frac{\partial\vartheta(t,x_t,r_t)}{\partial t} + \\ -r_t\vartheta(t,x_t,r_t) = 0 \\ \vartheta(T,x_t,r_t) = 1 \end{cases} \tag{86}$$

Proof. Let us assume $\vartheta(t,x_t,r_t) = B(t,T)$ with

$$B(t,T) = A^*(\theta)\,e^{-C^*(\theta)r_t}. \tag{87}$$

The Kolmogorov PDE becomes

$$\begin{cases} -\frac{1}{2}\sigma_x^2 x_t C^2(\theta)\,r_t C^{*2}(\theta)\,B(t,T)\left[1-C^{*2}(\theta)\,r_t\right] + \\ -a\left[b-x_t\right]C^*(\theta)\,B(t,T)\,r_t C(\theta) + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\,r_t^2 C^{*2}(\theta)\,B(t,T) + \\ -a^*(\theta)\left[b^*(\theta)-r_t\right]C^*(\theta)\,B(t,T) + \sigma_x^2 x_t C^2(\theta)\,r_t^2 C^{*2}(\theta)\,B(t,T) + \\ +A^{*'}(\theta)\,e^{C^*(\theta)r_t} - C^{*'}(\theta)\,r_t B(t,T) - r_t B(t,T) = 0 \\ \vartheta(T,x_t,r_t) = 1. \end{cases} \tag{88}$$

Since the PDE (88) is verified for all t, x_t and r_t, we can divide it into two equations, one dependent and one independent from x_t and r_t. Now the problem is to solve the DE system

$$\begin{cases} \frac{1}{2}\sigma^{*2}(\theta)\,C^{*2}(\theta) - \Psi(t)\,C^*(\theta) - C^{*'}(\theta) - 1 = 0 & C^*(0) = 0 \\ A^{*'}(\theta) - a^*(\theta)\,b^*(\theta)\,A^*(\theta)\,C^*(\theta) = 0 & A^*(0) = 1 \end{cases} \tag{89}$$

where $\sigma^*(t)$ and $\Psi(t)$ are

$$\sigma^*(\theta) = 2\sigma_x C(\theta)\sqrt{r_t x_t} \tag{90}$$

$$\Psi(\theta) = \frac{1}{2}\sigma_x^2 x_t C^2(\theta) + abC(\theta) - aC(\theta)\,x_t - a^*(\theta). \tag{91}$$

The first DE is a Riccati equation, and the second is a first order DE. Both equations have time-varying coefficient. We can then find the solution only by using numerical methods. □

Lemma 8. *The term structure of real interest rates is*

$$R(t,T) = -\frac{1}{\theta}\left[\ln A^*(\theta) - r_t C^*(\theta)\right]. \tag{92}$$

where $A^*(\theta)$ and $C^*(\theta)$ are the solutions of system (89).

Proof. The term structure of real interest rates can be derived from the relation between the price of zero coupon bond and the continuous real interest rate

$$B(t,T) = e^{-\theta R(t,T)}. \tag{93}$$

$\square$

6.3 The term structure of nominal interest rates

In this section we derive the equilibrium nominal spot interest rate, the analytical expression for the nominal zero coupon bond, and for the nominal term structure of interest rates.

Lemma 9. *The nominal spot interest rate is*

$$i_t = A(\theta)\, e^{C(\theta)x_t}\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{94}$$

where $A(\theta)$, $C(\theta)$ are (74) and (75), respectively.

Proof. Multiply (73) by (60). $\square$

Lemma 10. *The nominal zero coupon bond is*

$$N(t,T) = B(t,T)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{95}$$

where $B(t,T)$ is the solution of (88) .

Proof. Multiply the solution of (86) by (60). $\square$

Lemma 11. *The nominal term structure of interest rate is*

$$I(t,T) = R(t,T)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2(\beta+\mu_M^*)(\beta+\mu_M^*+2k_q\mu_q)}{(\beta+\mu_M^*)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{96}$$

where $R(t,T)$ follows from equation (92).

Proof. Multiply (92) by (60). □

From the expressions for (94), (95) and (96), we observe that the values of the rates for the nominal term structure are higher than those for the real variables if (60) is larger than one. This depends on μ_M^*, which is a function of ϕ_1. In particular, the nominal curve is above the real curve if

$$\mu_M^* < \mu_M^{*a} \text{ and } \mu_M^* > \mu_M^{*b} \tag{97}$$

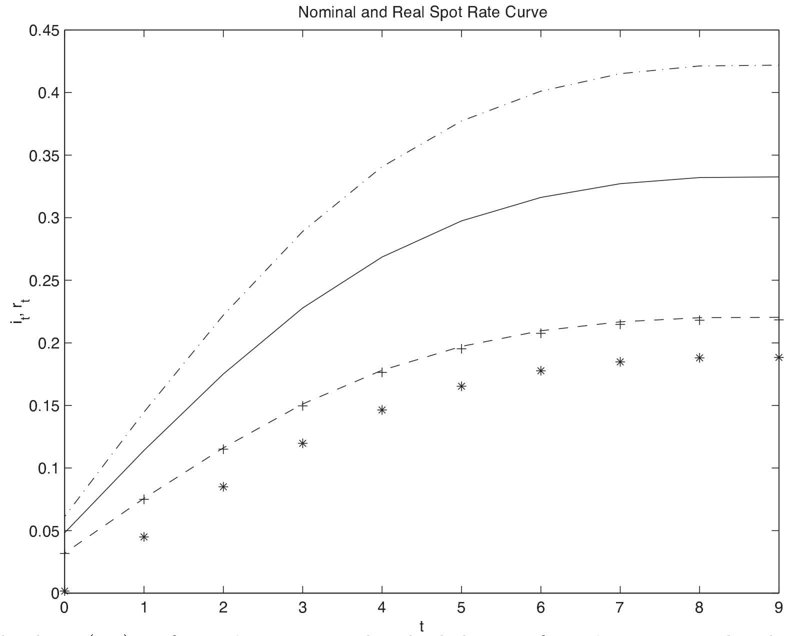
where μ_M^{*a} and μ_M^{*b} are

$$\begin{aligned}
\mu_M^{*a,b} &= \frac{-\Gamma \pm \sqrt{\Gamma^2+4M_t\phi q_t^2\Lambda}}{2M_t\phi q_t^2} && (98)\\
\Gamma &= 2M_t\phi q_t^2(\beta+k_q\mu_q) - Y_t(1-\phi) && (99)\\
\Lambda &= -Y_t(1-\phi)\beta - 2Y_t k_q\mu_q q_t\left(k_q+3\sigma_q^2\right)(1-\phi) && (100)\\
&\quad + M_t\phi q_t^2\beta(\beta+2k_q\mu_q).
\end{aligned}$$

7 Numerical simulations

For the simulation purposes we have chosen the following values for the parameters. The intertemporal substitution coefficient β has been set equal to 0.998. This is a common value assumed in both the financial economics and business cycle literature (for quarterly time interval). Moreover, the values for the other parameters has been set equal to what Balduzzi [2] has considered for a model with similar stochastic processes: $\kappa_q = 0.3$, $\mu_q = 0.1$, $\mu_Y = 0.2$, $\eta_Y = 0.4$, $\sigma_q = 0.1$, $\phi = 0.5$, $\mu_M = 0.2$. For the other parameters we have chosen values which are coherent from the economic point of view: $a = 0.45$, $b = 0.03$, $\sigma_x = 1.35$. The solution of the model depends also on the stochastic

Figure 1: The nominal and real spot curve



Legend: The line (−.) is for $\mu_M^* = 0.98$, the dark line is for $\mu_M^* = 0.55$, the dashed (−−) is for $\mu_M^* = 0.01$ and the crossed curve (++) is for $\mu_M^* = 0.0001$.

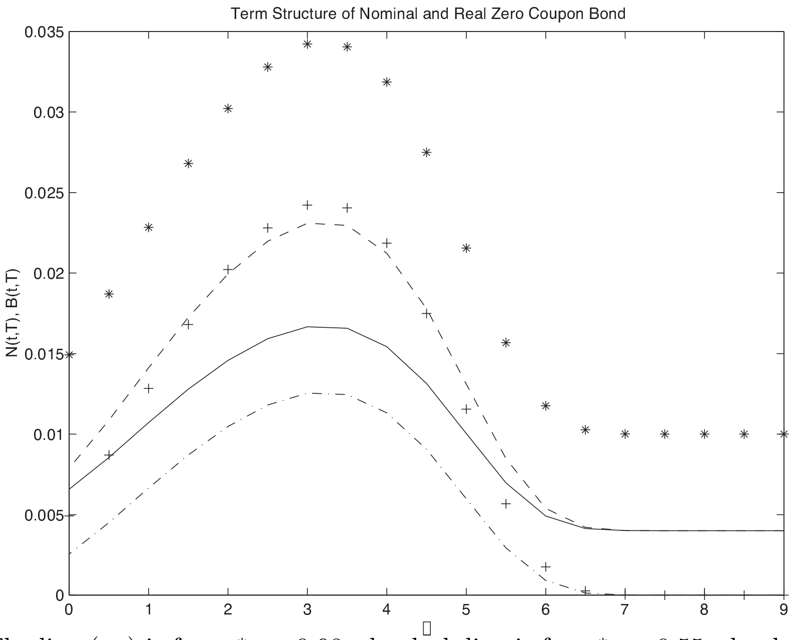
process for x, q, M and Y. We initialized these processes at the points $x_0 = 0.1$, $q_0 = 0.1$, $M_0 = 0.1$ and $Y_0 = 1$.

Figure 1 reports the nominal and real spot rate curve. The starred curve represents the real term structure. The other curves represent the nominal rates for different values of μ_M^*. The figure shows that, if the tax rate adjusts more strongly in response to changes of government liabilities (μ_M^* falls), the spot rate curve shifts down. When tax policy is very reactive, i.e. for small values of μ_M^*, the nominal curve appears to loose its sensitiveness to this parameter. Consistently with the economic theory discussed in this paper, the real curve lays below all the nominal curves.

In order to simulate the model for the term structure of zero-coupon bonds, we have obtained a closed form solution and made the additional assumption that the coefficients of equations (89) are time-independent. We are aware of the limitation of this choice. However this is needed to present the results in an intuitive way.

Figure 2 shows that the position in the plane of the term structure strictly depends upon the value assumed for μ_M^* and, consequently, upon the value of

Figure 2: The term structure of nominal and real zero-coupon bonds

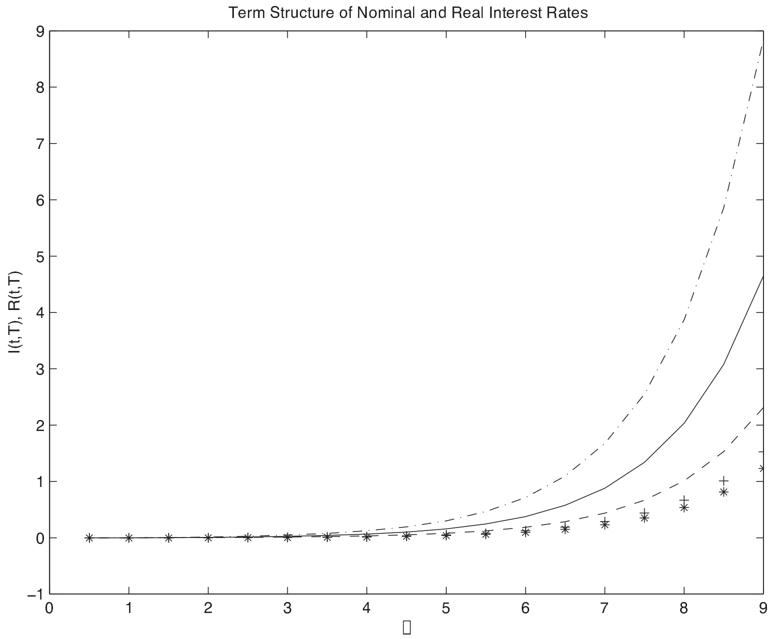


Legend: The line (−.) is for $\mu_M^* = 0.98$, the dark line is for $\mu_M^* = 0.55$, the dashed (−−) is for $\mu_M^* = 0.01$ and the crossed curve (++) is for $\mu_M^* = 0.0001$.

the tax rate ϕ_1. In particular, it is evident that if the tax rate decreases (i.e. if μ_M^* raises), the curve of the nominal term structure zero coupon bond shifts down, even if for very small values of μ_M^* (implied by a very high taxation policy) the curve appears to loose its sensitiveness to this parameter. The behavior of the curve is coherent with both the fiscal theory of the price level and the empirical evidence. In fact, if the tax sensitivity to government liabilities fall, the price demanded for newly-issued debt should drop in order to convince new subscribers to buy additional debt. The reason is that newly-issued debt without proper tax backing implies an inflationary risk for the future, thus causing a reduction of the nominal debt value already in the current period.

In Figure 3, we report the term structure of nominal and real interest rates. The position of the term structure in the plane depends strictly on the value assumed for μ_M^* and, consequently, upon the tax parameter ϕ_1. In particular, it is evident that if the tax sensitivity falls (i.e. if μ_M^* raises), the curve of the nominal term structure shifts upward. For low values of μ_M^*, implying a very reactive taxation policy, the curve appears to loose its sensitiveness to this parameter. This can be interpreted as follows. If the tax reaction to liabilities

Figure 3: The term structure of nominal and real interest rates



Legend: The line (−.) is for $\mu^*_M = 0.98$, the dark line is for $\mu^*_M = 0.55$, the dashed (−−) is for $\mu^*_M = 0.01$ and the crossed curve (++) is for $\mu^*_M = 0.0001$.

increases, it is possible to reduce the issue of public debt in order to finance the current position of the government. This calls for a lower interest rate and for enhanced credibility of the fiscal authority. In other words, monetary policy is not the only determinant of the term structure, but also fiscal policy is crucial. From an empirical viewpoint, the experience of Italy after the 1996-1997 episode of fiscal retrenchment represents a compelling example of shift in the position of the nominal term structure after a fiscal consolidation.

We should stress that also Figure 3 portrays a curve of real rates under the nominal curves. From an economic standpoint, the case where the nominal curve stays above the real one can be thought of as the result of a very tight fiscal policy that produces a deflationary equilibria. These types of equilibria are possible in the fiscal theory framework, as discussed by Sims [15]. This pattern can be interpreted also as arising from a contraction of consumer spending, and of aggregate demand, because an excessive level of fiscal pressure takes place.

8 Conclusion

In this paper we study a simple intertemporal model for the determination of the nominal and real term structure where the interaction between fiscal and monetary plays a key role. We investigate the relation between the term structure of interest rate and the fiscal theory of price level determination. In so doing, we move beyond the standard finance models where monetary and technological factors are the sole determinants of the term structure of interest rates.

A number of interesting avenues of future work can be considered. The model presented in this paper should be taken to the data. It would be important to use data on the supply of both public debt at various maturities and the money supply to capture better the correlation between nominal and real variables. An ongoing work considers the pricing of interest-rate derivatives, in particular options. This is likely to shed additional light on the influence that fiscal factors can play in shaping traders' views of bond pricing. Finally, a comprehensive derivation of measures of risk premia can be important to understand the fiscal sources of inflation risk.

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A Appendix

Proposition 12. *In the continuous time limit equilibrium, the real interest rate has the form*

$$r_t = \beta - \frac{Y_t u_{cc}}{u_c}\frac{1}{dt}E_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{1}{2}\frac{Y_t^2 u_{cc}}{u_c}var_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{m_t u_{cm}}{u_c}\frac{1}{dt}E_t\left\{\frac{dm_t}{m_t}\right\} \quad (101)$$

$$-\frac{1}{2}\frac{m_t^2 u_{cmm}}{u_c}var_t\left\{\frac{dm_t}{m_t}\right\} - \frac{Y_t m_t u_{ccm}}{u_c}cov_t\left\{\frac{dY_t}{Y_t}, \frac{dm_t}{m_t}\right\}. \quad (102)$$

Proof. A Taylor expansion around the equilibrium of first order condition (31) yields

$$u_c(Y_t, m_t)(1 + r_t\Delta t) = E_t\{(1 + r_t\Delta t)[u_c(Y_t, m_t) + u_{cc}(Y_t, m_t)\Delta Y_t +$$
$$+u_{cm}(Y_t, m_t)\Delta m_t + \frac{u_{ccc}(Y_t, m_t)}{2}(\Delta Y_t)^2$$
$$+\frac{u_{ccm}(Y_t, m_t)}{2}\Delta Y_t \Delta m_t + \frac{u_{cmm}(Y_t, m_t)}{2}(\Delta m_t)^2\Big]\Big\} + o(\Delta t)^{3/2}. \quad (103)$$

After collecting terms and re-arranging

$$r_t = \beta - \frac{1}{\Delta t}E_t\left\{\frac{u_{cc}Y_t}{u_c}\left(\frac{\Delta Y_t}{Y_t}\right) + \frac{u_{cm}}{u_c}m_t\left(\frac{\Delta m_t}{m_t}\right) + \frac{u_{ccc}Y_t^2}{2u_c}\left(\frac{\Delta Y_t}{Y_t}\right)^2 + \right.$$
$$\left.+\frac{u_{ccm}Y_t m_t}{2u_c}\left(\frac{\Delta Y_t}{Y_t}\right)\left(\frac{\Delta m_t}{m_t}\right) + \frac{u_{cmm}m_t^2}{2u_c}\left(\frac{\Delta m_t}{m_t}\right)^2\right\} + o(\Delta t)^{3/2}. \quad (104)$$

Thus, take the limit of (104) and apply Ito's multiplication rule, we obtain equation (102), after having recalled that

$$\frac{1}{dt}E_t\left\{\left(\frac{\Delta Y_t}{Y_t}\right)^2\right\} = var_t\left(\frac{\Delta Y_t}{Y_t}\right) \quad (105)$$

$$\frac{1}{dt}E_t\left\{\left(\frac{\Delta m_t}{m_t}\right)^2\right\} = var_t\left(\frac{\Delta m_t}{m_t}\right). \quad (106)$$

□

Corollary 13. *The real spot interest rate determined in equation (73) is also an equilibrium rate, for utility function parameter values ϕ such that this equation and (102) are equal. Consequently, this is true also for the nominal spot interest rate.*

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