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On Directional Immunization and Exact Matching

Werner Hürlimann[1]

Abstract

The rigorous version of Redington's theorem by Montrucchio and Peccati [12] is shown to be valid in a multivariate framework. It provides necessary and sufficient conditions for immunizing a fixed-income portfolio of assets and liabilities against a fixed non-parallel shift direction of the term structure of interest rates. As a consequence, immunization against all possible shift directions leads necessarily to an exact matching strategy.

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1 Introduction

In recent years important advances on immunization theory have been achieved. Several authors have undertaken the task to extend the original model

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[1] FRSGlobal Switzerland AG, e-mail: whurlimann@bluewin.ch

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by Redington [14] to take into account more realistic real-world assumptions. Fisher and Weil [6] have worked with a non-flat term structure of interest rates and have shown how to immunize a single liability for parallel shifts in the term structure. This approach was generalized to multiple liabilities by Shiu [21] who also obtained interesting connections with Linear Programming (see also [19], [20], [22]). Extension of this approach and a remarkable rigorous statement of Redington's theorem are discussed in Montrucchio and Peccati [12]. The author has derived in [9] some main immunization results based on some well-known equivalent characterizations of the stop-loss order by equal means, also called convex order. In a special case, the conditions by Fong and Vasicek [7], [8], and Shiu [21], are extended to a necessary and sufficient condition for immunization under arbitrary convex shift factors of the term structure of interest rates. Based on a linear control problem with the Shiu measure as objective function, the bounds by Uberti [21] on the change in portfolio value of Shiu decomposable portfolios under alpha-convex and convex-beta shift factors have also be analyzed in detail. Some of the latter results have also been extended to the immunization of economic cash-flow products as considered first in Costa [4] (for this consult [10], Sections 7 and 8). A multivariate model to deal with non-parallel shifts of the term structure has been motivated and developed in a series of papers by Reitano [15] to [18].

All of the above concerns (mainly) deterministic immunization theory. To take into account also interest-sensitive cash flows, unknown random times of payments, and other stochastic modeling assumptions, further extensions are needed. Several authors have already worked along this line, e.g. Boyle [2], Albrecht [1], Castellani, DeFelice and Moriconi [3], Shiu [23] and Munk [13].

A scrutiny look at the recent work in the deterministic framework shows that ultimate accomplishment in this direction of study is still not yet achieved. In this note a rigorous multivariate Redington theorem is formulated and proved for the case of non-parallel shifts of the term structure. This result extends Proposition

8 in Montrucchio and Peccati [12] applying the technic of multivariate immunization theory as developed by Reitano [18].

2 Multivariate Duration and Convexity Analysis

In this communication we restrict ourselves to the financial context of fixed-income portfolios including investment in bonds. The goal of immunization theory is to protect the investment value of a portfolio of known asset and liability payments against interest rate fluctuations.

To describe the Term Structure of Interest Rates (TSIR) over the time horizon $[0,T]$, we work in a discrete time setting and assume throughout the time unit is chosen such that all payments are made at times an integer multiple of the time unit. For $k=1,...,T$, let R_k be the random rate of return over the period $[k-1,k)$, and let $\delta_k = \ln\{E[R_1...R_k]\}/k$ be the one-periodic instantaneous expected rate of return over the period $[0,k)$. In this notation $i_k = \exp(\delta_k)-1$ is the one-periodic spot rate over the period $[0,k)$ and $d_k = \exp(-k\delta_k)$ is the discount factor used to calculate the present value of financial quantities to be paid at time k. The TSIR is assumed to be described by the vector $\delta = (\delta_1,...,\delta_T)$, which is considered as an equivalent of the yield curve as given on the market place of fixed income securities.

Arbitrary shifts $\varepsilon \cdot \omega$ of the TSIR can occur in the direction $\omega = (\omega_1,...,\omega_T) \neq 0$ with a magnitude of ε, a real number. To normalize direction vectors one uses the standard Euclidean norm $|\omega|^2 = \sum_{j=1}^{T} \omega_j^2$. We assume that $|\omega|^2 = T$ is the norm of a classical parallel shift $\omega = (1,...,1)$. Further, it is always assumed that the yield vector changes from δ to $\delta^* = \delta + \varepsilon \cdot \omega$ immediately after time 0, and remains at this level throughout the period. The meaning and practical use of this assumption is explained in Reitano [18], section II.B.

Given a fixed stream $\{P_k\}$ of positive payments, where P_k is to be paid at time k, its price or present value is described by the multivariate function

$$P(\delta) = \sum_{k=1}^{T} P_k \exp(-k\delta_k). \tag{2.1}$$

One is interested in the price $P(\delta^*)$ whenever δ changes to $\delta^* = \delta + \varepsilon \cdot \omega$. A second order multivariate Taylor expansion describes the price adjustments to be made as follows:

$$P(\delta + \varepsilon \cdot \omega) \approx P(\delta) \cdot \left(1 - \varepsilon \cdot \sum_j D_j(P)\omega_j + \tfrac{1}{2}\varepsilon^2 \cdot \sum_j \sum_k C_{jk}(P)\omega_j \omega_k \right). \tag{2.2}$$

In this approximation $D_j(P)$ and $C_{jk}(P)$ are partial durations and partial convexities, which are first and second order sensitivity measures to a price change due to a shift in the TSIR. In a general context and provided $P(\delta) \neq 0$ they are defined by the partial derivatives

$$D_j(P) = -\frac{1}{P(\delta)} \frac{\partial P(\delta)}{\partial \delta_j}, \quad C_{jk}(P) = -\frac{1}{P(\delta)} \frac{\partial^2 P(\delta)}{\partial \delta_j \partial \delta_k}, \quad j,k = 1,\ldots,T. \tag{2.3}$$

Under the assumption (2.1) one has

$$D_j(P) = \frac{jP_j \exp(-j\delta_j)}{P(\delta)}, \quad C_{jk}(P) = \begin{cases} 0, & k \neq j \\ \dfrac{j^2 P_j \exp(-j\delta_j)}{P(\delta)}, & k = j. \end{cases} \tag{2.4}$$

Besides these partial sensitivity measures, one considers also directional measures, namely notions of directional duration and directional convexity derived from the first and second order directional derivatives of $P(\delta)$ in the direction of ω:

$$\begin{aligned} D_\omega(P) &= -\frac{d_\omega P(\delta)}{P(\delta)} = \omega \circ D(P) = \sum_j \omega_j D_j(P), \\ C_\omega(P) &= \frac{d^2{}_\omega P(\delta)}{P(\delta)} = \omega^T \circ C(P) \circ \omega = \sum_j \sum_k \omega_j \omega_k C_{jk}(P), \end{aligned} \tag{2.5}$$

where $D(P), C(P)$ denote a total duration vector, respectively a total convexity matrix. In this multivariate pricing model, the classical model of Redington with parallel yield curve shifts and flat yield curve is recovered as the special case $\omega = (1,...,1)$ and $\delta = (\delta,...,\delta)$. In this case the directional measures specialize to the more traditional measures of (modified) duration and convexity:

$$D_{(1,...,1)}(P) = \sum_j D_j(P) = -\frac{P'(\delta)}{P(\delta)}, \quad C_{(1,...,1)}(P) = \sum_j \sum_k C_{jk}(P) = \frac{P''(\delta)}{P(\delta)}. \tag{2.6}$$

Under the assumption (2.1), the directional duration and convexity measures have the following probabilistic interpretation, which will be used later on. Consider the discrete random variable X_ω defined by

$$\Pr(X_\omega = j\omega_j) = \frac{P_j \exp(-j\delta_j)}{P(\delta)}, \quad j = 1,...,T. \tag{2.7}$$

Then one has

$$D_\omega(P) = E[X_\omega], C_\omega(P) = E[X_\omega^2]. \tag{2.8}$$

Furthermore, the variance of X_ω defines a notion of directional dispersion defined by

$$M_\omega^2(P) = Var[X_\omega] = C_\omega(P) - D_\omega(P)^2. \tag{2.9}$$

3 A Multivariate Version of Redington's Immunization Theorem

In Montrucchio and Peccati [12], Proposition 8, a rigorous Redington immunization theorem has been formulated and proved. To immunize a portfolio of fixed assets and liabilities against interest rate fluctuations in the case of parallel shifts of the TSIR, it suffices, first, to equate the duration of the assets to the duration of the liabilities, second, to require more dispersion for the assets than for the liabilities and, third, to restrict the amplitude of the interest rate shock to a well-defined small number. In this communication we show that this result is a special case of a rigorous Redington directional immunization theorem valid for shifts of the TSIR in a given fixed direction. As a consequence, the immunization of a portfolio of fixed assets and liabilities against arbitrary shifts of the TSIR is only possible through an exact matching strategy.

Given is a fixed-income portfolio consisting of a stream $\{L_k\}$ of liability payments to be paid at future times $k = 1,...,T$. The liabilities are funded by a stream $\{A_k\}$ of asset cash flows, where the cash inflow A_k occurs at time k. The time ranges of the two payment streams are denoted by

$$[m_A, n_A] \quad \text{an} \quad [m_L, n_L],$$

where

$$m_A = \min\{k : A_k \neq 0\}, \qquad n_A = \max\{k : A_k \neq 0\},$$

$$m_L = \min\{k : L_k \neq 0\}, \qquad n_L = \max\{k : L_k \neq 0\}.$$

Suppose that the current TSIR is described by a function $\delta(t), t \in [0,T]$, that represents an instantaneous rate of return or force of interest. A change in the TSIR occurs always immediately after time 0 through an interest rate shock function $\varepsilon(t), t \in [0,T]$, such that $\delta(t)$ changes to $\delta(t) + \varepsilon(t)$. The link with the vectors δ and $\varepsilon \cdot \omega$ of section 2 is given by difference operators:

$$\delta(t) = \Delta\{k\delta_k\} = (k+1)\delta_{k+1} - k\delta_k, \quad \varepsilon(t) = \varepsilon \cdot \Delta\{k\omega_k\}, \quad t \in [k, k+1). \tag{3.1}$$

For example, a classical parallel shift $\omega = (1,...,1)$ has a constant shock function $\varepsilon(t) = \varepsilon$. In the multivariate framework, we assume that $\delta(t), \varepsilon(t)$ are of the form (3.1). The prices of the individual asset and liability payments with respect to the yield curve vector δ are denoted

$$a_k = a_k(\delta) = A_k \exp(-k\delta_k), \quad \ell_j = \ell_j(\delta) = L_j \exp(-j\delta_j), \quad k, j = 1,...,T. \tag{3.2}$$

The total prices are denoted by $A = A(\delta) = \sum_k a_k$, $L = L(\delta) = \sum_j a_j$. We assume that the equivalence principle holds. That is the stream of liability payments is fully funded at time 0, and its price is given by

$$P = P(\delta) = A(\delta) = L(\delta). \tag{3.3}$$

The discount factor shock function of an interest rate shock $\varepsilon \cdot \omega$ of magnitude ε in direction $\omega \neq 0$ is denoted by

$$f_{\omega}(t) = \exp\left\{-\int_{0}^{t} \varepsilon(u)du\right\}. \tag{3.4}$$

If $t = k + u, \quad 0 \leq u < 1, \quad k \in \{0,...,T-1\}$, then one has using (3.1)

$$f_{\omega}(k+u) = \exp\{-\varepsilon \cdot (k\omega_k + u \cdot \Delta\{k\omega_k\})\}. . \tag{3.5}$$

In the special case of parallel shifts, we write simply $f(t)$ instead of $f_{\omega}(t)$, and

$$f(t) = \exp(-\varepsilon \cdot t), \quad t \in [0,T]. \tag{3.6}$$

The present value of the portfolio as function of the yield curve vector δ is denoted $G(\delta)$. By assumption (3.3) one has $G(\delta) = A(\delta) - L(\delta) = 0$. Whenever the TSIR changes from δ to $\delta + \varepsilon \cdot \omega$, one has

$$\begin{aligned} G(\delta + \varepsilon \cdot \omega) &= A(\delta + \varepsilon \cdot \omega) - L(\delta + \varepsilon \cdot \omega) \\ &= \sum_k a_k f_{\omega}(k) - \sum_j \ell_j f_{\omega}(j) = \sum_k a_k f(k\omega_k) - \sum_j \ell_j f(j\omega_j) \end{aligned} \tag{3.7}$$

To immunize the portfolio in direction ω (notion of *directional immunization*), one requires the condition $G(\delta + \varepsilon \cdot \omega) \geq 0$.

Let us state and prove the following main result, which is a direct generalization of Proposition 8 in Montrucchio and Peccati [12].

Theorem 3.1 *Necessary* conditions to achieve immunization in direction ω for small shocks $\varepsilon \cdot \omega$ of magnitude ε are

$$D_{\omega}(A) = D_{\omega}(L) \qquad \text{(directional duration principle)}, \tag{3.8}$$

$$M_{\omega}^2(A) \geq M_{\omega}^2(L) \qquad \text{(directional dispersion principle)}. \tag{3.9}$$

Furthermore, if the magnitude ε of the shock belongs to the interval

$$\left[\frac{1}{(n_L - m_A)} \ln\left\{\frac{M_{\omega}^2(L)}{M_{\omega}^2(A)}\right\}, \frac{1}{(n_A - m_L)} \ln\left\{\frac{M_{\omega}^2(L)}{M_{\omega}^2(L)}\right\}\right], \tag{3.10}$$

then the conditions (3.8) and (3.9) are also *sufficient* to achieve directional immunization.

Proof. We observe that the proof of Montrucchio and Peccati [12] carries over without difficulty. In fact, it suffices to work with discrete random variables of the kind defined in (2.7), and to consider the last relation in (3.7), which depends directly on the function $f(t)=\exp(-\varepsilon\cdot t)$ used in that proof. For convenience, we provide the main steps.

Given two real numbers α and β on an interval I, one says that a function f(t) is α-convex on I, respectively convex-β on I, if $f(t)-\frac{1}{2}\alpha t^2$, respectively $f(t)-\frac{1}{2}\beta t^2$, is convex on I, respectively concave on I. First of all, one sees that $f(t)=\exp(-\varepsilon\cdot t)$ is α_A-convex and convex-β_A on the interval $[m_A,n_A]$, and $f(t)$ is also α_L-convex and convex-β_L on $[m_L,n_L]$. Appropriate values of α_A, α_L, β_A, β_L are as follows: if $\varepsilon\geq 0$

$$\begin{aligned}\alpha_A&=\inf\{f''(t): t\in[m_A,n_A]\}=\varepsilon^2\exp(-\varepsilon\cdot n_A),\\ \beta_A&=\sup\{f''(t): t\in[m_A,n_A]\}=\varepsilon^2\exp(-\varepsilon\cdot m_A),\end{aligned}\tag{3.11}$$

and, if $\varepsilon\leq 0$

$$\alpha_A=\varepsilon^2\exp(-\varepsilon\cdot m_A),\quad \beta_A=\varepsilon^2\exp(-\varepsilon\cdot n_A).\tag{3.12}$$

Similar values are obtained for α_L and β_L. Consider now discrete random variables X_ω and Y_ω defined as in (2.7) by

$$\Pr(X_\omega=j\omega_j)=\frac{a_j}{A(\delta)},\quad \Pr(Y_\omega=j\omega_j)=\frac{l_j}{L(\delta)},\quad j=1,...,T.\tag{3.13}$$

From Jensen's inequality and the above convexity properties one obtains the following two sets of inequalities :

$$\begin{aligned}E[f(X_\omega)]&\geq f(E[X_\omega])+\tfrac{1}{2}\alpha_A Var[X_\omega],\\ E[f(Y_\omega)]&\leq f(E[Y_\omega])+\tfrac{1}{2}\beta_L Var[Y_\omega],\end{aligned}\tag{3.14}$$

$$\begin{aligned}E[f(X_\omega)]&\leq f(E[X_\omega])+\tfrac{1}{2}\beta_A Var[X_\omega],\\ E[f(Y_\omega)]&\geq f(E[Y_\omega])+\tfrac{1}{2}\alpha_L Var[Y_\omega].\end{aligned}\tag{3.15}$$

Subtracting these inequalities in pairs using the probabilistic relations (2.8), (2.9) as well as (3.3) and (3.7), one obtains the bounds

$$
\begin{aligned}
&\frac{1}{P}G(\delta+\varepsilon\omega)\geq f(D_\omega(A))-f(D_\omega(L))+\tfrac{1}{2}\left(\alpha_A M_\omega^2(A)-\beta_L M_\omega^2(L)\right),\\
&\frac{1}{P}G(\delta+\varepsilon\omega)\leq f(D_\omega(A))-f(D_\omega(L))+\tfrac{1}{2}\left(\beta_A M_\omega^2(A)-\alpha_L M_\omega^2(L)\right),
\end{aligned}
\tag{3.16}
$$

from which one deduces the necessary condition for directional immunization

$$
f(D_\omega(A))-f(D_\omega(L))\geq\frac{1}{2}\left(\alpha_L M_\omega^2(L)-\beta_A M_\omega^2(A)\right),\tag{3.17}
$$

and the sufficient condition

$$
f(D_\omega(A))-f(D_\omega(L))\geq\frac{1}{2}\left(\beta_L M_\omega^2(L)-\alpha_A M_\omega^2(A)\right).\tag{3.18}
$$

In case $\varepsilon\geq 0$ the necessary condition (3.17) can be rewritten as

$$
\exp\{-\varepsilon D_\omega(A)\}-\exp\{-\varepsilon D_\omega(L)\}\geq\frac{1}{2}\left\{\varepsilon^2\exp(-\varepsilon n_L)M_\omega^2(L)-\varepsilon^2\exp(-\varepsilon m_A)M_\omega^2(A)\right\}.
$$

Dividing by ε and taking the limit as ε goes to 0+, one obtains

$$
D_\omega(L)-D_\omega(A)\geq 0.
$$

A similar argument for $\varepsilon\leq 0$ shows that

$$
D_\omega(L)-D_\omega(A)\leq 0.
$$

Therefore the directional duration principle (3.8) holds. Now the necessary condition reads

$$
\varepsilon^2\exp(-\varepsilon n_L)M_\omega^2(L)\leq\varepsilon^2\exp(-\varepsilon m_A)M_\omega^2(A).
$$

Dividing by ε^2 and taking the limit as ε goes to 0, one obtains the directional dispersion principle (3.9). Consider now the sufficient condition (3.18). If $\varepsilon\geq 0$ one has

$$
\exp(-\varepsilon m_L)M_\omega^2(L)\leq\exp(-\varepsilon n_A)M_\omega^2(A).
$$

Taking logarithms one obtains the upper bound of the interval (3.10). The inequality $n_A-m_L>0$ follows from the directional duration principle (3.8). Through similar calculation for $\varepsilon<0$ we obtain the lower bound in (3.10). □

4 Conclusion

To conclude this communication, a few remarks are in order. The necessary conditions (3.8) and (3.9) imply "local immunization of the surplus at all times $k \geq 0$ in the direction of ω on the yield curve vector δ" in the sense of Reitano [18], section II.B. This directional Redington immunization result is a particular case of Proposition 3 in Reitano [18], section IV, and is equivalent to the necessary part of the above Theorem. Passing to a non-directional immunization model, that is immunization against all possible shift directions from δ, leads necessarily to exact matching programs in the context of asset/liability management with fixed-income portfolios. Indeed the validity of the directional duration principle $D_\omega(A) = D_\omega(L)$ for all $\omega \neq 0$ implies equality for all partial durations: $D_j(A) = D_j(L),\ j = 1,...,T$. Using formula (2.4) this means that $A_j = L_j,\ j = 1,...,T$, which is exact matching. Finally, note that algorithms for exact matching are for example found in the book by Elton and Gruber [5], chap. 19, appendix B, as well as in the paper by Kocherlakota, Rosenbloom and Shiu [11].

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