

The Dynamics of Market Share's Growth and Competition in Quadratic Mappings

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Abstract

This paper shows that the observed output of any market, placed within the confine of a quadratic map, can characterize the state of that market. Such an approach explains the process of market share's growth and its pitfalls, the consequences of broken symmetry of scaling, as well as the limits of firms' competition for market shares.

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1 Introduction

The literature on market share's dynamics focuses mainly on spatial competition with quadratic costs, duopolistic strategies, and competition in attraction models, but much less so on the growth of market share and its pitfalls.

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This might be due to the inability of these approaches to capture the full complexity of growth dynamics. Yet, there exists, at this juncture, a wide consensus on the fact that market share's growth dynamics are non-linear and complex. In fact, it has been clearly demonstrated that most processes in economics and finance are better characterized by fractal attractors [1], [2], [3], [4] [5], [6]. It would seem, therefore, that many important but yet hidden insights into process of market's share growth could be gained from tackling the notion of complexity head on. And the simplest way to proceed is to turn to quadratic mappings, i. e., to the Logistic parabola in particular.

Herein, a quadratic mapping is taken to be a type of mappings that uses the previous value of a process raised to the power of two, but which may or may not have a closed-form solution. We opted for the Logistic map because it has one. Additionally, as a recurrence relation of degree two, it is the simplest example of how complex behavior can arise from very simple non-linear relations. More succinctly put, the Logistic parabola is a prototypical "strange" attractor. This might explain why it has found applications in fields as diverse as semi-conductor analyses, thermodynamics, biology, medicine, ecology, and even socio-economics. Moreover, on a higher explanatory level, the Logistic map lays bare, albeit empirically, the consequences of breaking the fundamental concept of "symmetry", which may simply be defined here as an *invariance to change*. To wit: The sharp distinction between "monofractality" and "multifractality" (or self-affinity), revealed by the logistic map arises precisely from a *broken symmetry of scale* made operational as a *broken symmetry of translation of equilibria* on its linear term (*vide infra*).

This paper will draw on these rich insights. But for tractability and completeness, it will first briefly review the salient features of the Logistic parabola in order to show that any market can be embedded and analyzed in the *non-convex set* bounded by the "hypograph" of the envelop map (f^{\max}) and the "epigraphs" of f^{\min} and that of the linear term of the map. In so doing, the restrained growth of market share, the impacts of competition, firms' efficiency, and even the loss of information of a dynamic system evolving in time can easily be characterized.

2 Theoretical Preliminaries

Consider a family of mappings (or Iterated Function Systems):

$$F = \{f^{\max}, f^m(f^i, f^j)\}, i \in n, j \in n, i \neq j, \quad (1)$$

and a choice variable $s_i = Q_i/Q$, where $f^{\max}: (0, 1) \rightarrow (0, 1)$ is the "envelop" map whose hump occurs at $s = 1/2$. f^m (or f^{\min}) is the observable map of any market; f^i and f^j describe the mappings of firm i and firm j , respectively, operating in the

observed market whose total output is $Q = 1$. It follows that $f^{\max} > f^m \geq f^{\min}$, and $(f^i, f^j) \in f^m$.

It is convenient to begin the analysis with a general description (f^{\max}) of the restrained growth process of s . That is,

$$f^{\max} = K^{\max} s_t - K^{\max} h s_t^{\eta+1}, \text{ where } K^{\max} > 1. \tag{2}$$

This implies that market share grows at a fixed rate determined by K^{\max} , but at the cost ($h s_t^{\eta+1}$) of generating that discrete level of market share in time t , evaluated in terms of market share. Another way of describing the growth process depicted by the above equation is to suppose that the *net* level of s_t grows at a rate determined by K^{\max} . Then,

$$f^{\max}(s) = s_{t+1} = K^{\max} s_t (1 - h s_t^{\eta}). \tag{3}$$

In other words, the growth factor falls to zero as $s_t \rightarrow 1$. Since $f^{\max}(\cdot) \in (0, 1)$, the following restrictions are in order:

$$0 < \eta \leq 2, \text{ and } h > [1/(\eta + 1)].$$

Denoting s^- , K^{\max} , and s^* as the mean, the growth factor, and the equilibrium value, respectively, we have:

$$s^- = \{1/[h(\eta + 1)]\}^{1/\eta}, K^{\max} = \{1/[s^-(1 - h s^{-\eta})]\}, s_{t^*} = [(K^{\max} - 1) / h K^{\max}].$$

It can be seen that if $h = 1$, $\eta = 1$, $s^- = 1/2$, $K^{\max} = 4$, and $s^* = 3/4$. Then (3) would be reduced to a modified version of Jean-François Verhulst's growth equation, a. k. a. the Logistic parabola. It should also be stressed at this point that s^* to be stable must satisfy: .

$$df^{\max}/ds = K^{\max} [1 - h s_t^{\eta}(\eta + 1)] \in [-1, 1]; \tag{4}$$

whether in the envelop map or in its iterates, and $df/ds = 0$ represents a *super-stable equilibrium*.

2.1 The Dynamics of f^{\max}

The dynamics of (3) may be summarized as follows: Let $N(B)$ and $\varphi_t(s) \subset N(B)$ be an open neighborhood and the flow of (3), respectively. If $\varphi_t(\cdot) \subset N(B)$ at $t \geq 0$ and $\varphi_t(\cdot) \rightarrow B$ as $t \rightarrow \infty$, then B is a compact hyperbolic attractor for f^{\max} . Further, denote the locally stable and unstable manifolds for a small neighborhood of B as S and U , respectively. Then by letting points in S flow forward in time while points in U flow backward in time, the globally stable and unstable manifolds of f^{\max} are:

$$M^s = \bigcup_{t \geq 0} \varphi_t(B), \text{ and } M^u = \bigcup_{t \leq 0} \varphi_t(B). \tag{5}$$

These manifolds being unique and invariant under the flow, it then follows that,

$$\forall s \in M^s, \lim_{t \rightarrow \infty} \varphi_t(\cdot) = B, \quad \text{and} \quad \forall s \in M^u, \lim_{t \rightarrow -\infty} \varphi_t(\cdot) = B.$$

Moreover, we define:

$$\Gamma^+(s) = \{s \in \mathfrak{R} \mid s = \varphi_t(s_0), t \geq 0\}, \quad \text{and} \quad \Gamma^-(s) = \{s \in \mathfrak{R} \mid s = \varphi_t(s_0), t \leq 0\}$$

as an orbit or a positive half trajectory and a negative half trajectory through s_0 , respectively, such that $\Gamma = \Gamma^+ \cup \Gamma^-$

Then:

$$M^s(\Gamma^+) = \bigcup_{t \geq 0} \varphi_t(S(\Gamma)) \quad \text{and} \quad M^u(\Gamma^-) = \bigcup_{t \leq 0} \varphi_t(U(\Gamma)). \tag{6}$$

To simplify further and denoting λ as the Lyapunov characteristic exponent (LCE), we posit:

$$\Gamma^+ \in M^s_{(\lambda < 0)}, \quad \leftrightarrow \quad \Gamma^0_{np} \in M^o_{(\lambda = 0)}, \quad \Gamma^- \in M^u_{(\lambda > 0)} \quad \leftrightarrow \quad \Gamma^0_{np} \in M^o_{(\lambda = 0)};$$

That is, Γ^+ , Γ^- , Γ^0 represent stable or periodic, unstable, and non-periodic orbits, respectively; orbits are either stable or unstable except at aperiodicity where all orbits become non-periodic.

For further ease of exposition, consider the following definitions:

Definition 1 [7]. *B is a strange attractor if it contains a countable subset of periodic orbits (Γ_p), an uncountable subset of non-periodic orbits (Γ_{np}), and a dense orbit (Γ_d).*

Definition 2. *If W is a subset of B, W is dense in B if for every point $b \in B$ and a $\delta > 0$, there is a point $w \in W$ such that $(b - w) < \delta$.*

Definition 3. [8]. *If two close points $b_i, b_j \in N(B)$ at $t \geq 0$ becomes exponentially distant as $t \rightarrow \infty$, then Sensitive Dependence on Initial Conditions (SDIC) exists.*

Thus, B is hyperbolic if $\exists (\Gamma^+, \Gamma^-)$; B is strange if $\exists (\Gamma_p, \Gamma_{np}, \Gamma_d)$ by Definition 1 and Definition 2; and B is chaotic if $\exists (\Gamma^+, \Gamma^-)$, SIDC) by Definition 1 – Definition 3.

Suppose for moment that $N(B) \in \mathfrak{R}^3$. Suppose further that all orbits that begin in $N(B)$ at time $t \geq 0$ do so in an inward direction, wonder about $N(B)$, and end up in $\varphi_\tau(\cdot)$ (which is an image set of $U(B)$) at time $t + \tau$ under the transformation of (3). Then, it can be shown that the volume of $\varphi_\tau(\cdot)$ shrinks to zero [9], [10]. This then means that B is an attractor that comprises two subsets of points of zero volume. Eq. (3) is therefore a dissipative system containing a strange attractor

$B = \bigcap_{\tau \geq 0} \varphi_{\tau}(N(B))$. But, due to the loss of energy (information), shrinking volume turns B into a “thin” set containing the interleaved subsets of points of zero volume. These interleaved subsets intersect. Trajectories (Γ^+, Γ^-) , on the other hand, do not, but may move from one subset to another as they circulate.

Here we suppose that $N(B) \in \mathfrak{R}^3$ for concreteness. If $f^{\max} \in \mathfrak{R}^2$, then $N(B) \in \mathfrak{R}^{2_{++}}$ is a positive area, and the above discussion more properly refers to points of zero area; and on the unit interval, it refers to points of zero dim.

2.2 Characterizing f^{\max}

For the complete characterization of f^{\max} the reader is referred to Figure 1 and Table 1 at the end of the paper. Figure 1 uses the Hausdorff dimension (D_0) for a number of reasons. First, D_0 is the starting point in the determination of the singularity spectrum of f^{\max} ; as we will see in the last section, the spectrum may become handy to distinguish between two regions of persistence. Second, D_0 accommodates both integer and non-integer values. Thirdly, D_0 shows that points and unions of points of zero area have zero dimensions. Finally, non-periodic orbits mean that there are points on the attractor that are visited only once, and there are points that are never visited. It so happens that D_0 is non-probabilistic measure of how orbits fill up the available space. Therefore, for our purpose, D_0 is much more efficient than both the topological and box-counting dimensions.

At this juncture, it is clear that variations of the growth factor generate sub-maps and intervals of interest. The entries of Table 1 are self explanatory but for two brief comments. The Benoit_{TM} software used to compute D_0 does not pick up certain theoretical critical values of K^{\max} , except that at $K^{\max} = 3.44$ where changes begin to occur. Surprisingly, the first bifurcation at $K^{\max} = 3.23$ appears in the monofractal regime. The second occurs during the first enlargement of B . Another case in point is the exact place where the process becomes aperiodic. There, theoretical and computed values are at variance. Grassberger [11] has determined theoretically and analytically that the Hausdorff dimension of quadratic maps at aperiodicity ($K^{\max} = 3.5699\dots$) to be $D_0(\in \mathfrak{R}) = 0.5381 \pm 0.002$. But Table 1 indicates aperiodicity at $K^{\max} = 3.59\dots$ with a $D_0(\in \mathfrak{R}) = 0.58\dots$, probably due to instrumental noise.

For the present purpose, however, it is worth underlining two other essential points. That is, the broad division of the K -continuum and the clear distinction between monofractality and multifractality. Referring to Figure 1, it is easily seen that monofractality (or self-similarity) begins at $K^{\max} = 3.0\dots$ with a power spectrum $\beta = 2.84$. In the interval $3.0 < K \leq 3.44$, there is *persistent monofractality* and the size of the attracting set B is rather small. Beyond that interval, the process becomes multifractal (or self-affine). The flip to multifractality is accompanied by significant enlargements of B beginning at $K^{\max} = 3.44 \dots$ and again at $3.60 \dots$. Computed values show that from $3.48\dots$ to 3.59

...the process is mildly multifractal. However, over the interval $3.59 < K^{\max} \leq 3.84$, the process is *multifractal but severely anti-persistent*. In that region, there are a few stable orbits (including a mega-cycle at $K^{\max} = 3.74$) and an infinite number of unstable orbits coexisting with chaotic K-values. These values are densely interwoven with non-chaotic K-values, but they do not form intervals and they are all very sensitive to noise. Whereas between 3.84 and 4.0, the process remains *multifractal but persistent*. The interesting thing here to note is that in that region usually characterized as chaotic, the subset of stable orbits is not empty.

Another point worth stressing is that the attractor becomes strange ($K^{\max} = 3.59\dots$). We attribute strangeness to the fact that B is an invariant set whose tangent space diverges into stretching and contracting directions. Previously, each point b in B was either in the stable manifold which consists of all points that are positively asymptotic to b, or in the unstable manifold consisting of all points that are negatively asymptotic to b. At $b \in B$, M^s was tangent to the contracting direction, and M^u was tangent to the stretching direction. But at aperiodicity, the stretching and the shrinking, caused by stable orbits migrating toward that accumulation points, convert both M^s and M^u into M^o , thereby making the attractor strange. Incidentally, it is hypothesized that at aperiodicity, the attractor should appear as a Cantor point set. However, our software shows a $D_0(\epsilon \mathcal{R}) = 0.58 \dots$ instead of 0.6309 for the Cantor point set.

In short, the characterization of f^{\max} may be summarized as follows: Upon variations of K, the invariant set B changes its nature. To wit: From $K > 1$ and $K < 3.23$, B is a fixed-point attractor. Between $K = 3.23$ and $K < 3.59$, $\exists (M^s, M^u, \text{stretching})$. At $K = 3.59$, M^s and M^u change to M^o , but there is no chaos as defined here. Between $K = 3.60$ and $K = 3.84$, M^o changes back to M^s, M^u and B undergoes a second stretching. Between $3.84 < K < 4$, $\exists (M^s, M^u, \text{folding of B, Sensitivity to parameters (STP), and SIDC})$. Finally at $K = 4$, there is chaos proper, $M^s \rightarrow M^u$, i. e., $\forall \Gamma^- \rightarrow -\infty$.

At $K^{\max} \approx 3.45$ and $K^{\max} \approx 3.60$, etc., the attracting set undergoes enlargements, while at $K^{\max} \approx 3.85$, it undergoes a shrinking. Between $K = 3.23$ and $K = 3.59$, orbits go from M^s to M^u and back to M^s after bifurcations. Between $K = 3.60$ and $K < 4$, Γ^+ venture into chaotic K-values and back, etc. We attribute these to the phenomenon called *intermittency*. We detect two types of intermittency. In the most likely due to phase shifting at $K = 3.34, K = 3.50$, etc. or to stable orbits venturing into chaotic intervals before returning to the previously stable orbits. Anyway, more will be said about intermittency in the conclusions.

The last point to stress at this juncture is the fact that the properties of quadratic maps are generic. For example, by varying the parameters h and η in (3), various quadratic maps can be generated; differences between them arise only from differences in intervals of the growth parameter K.

3 Characteristics of Actual Markets

In Eq. (1), f^m stands for an observable market located between the hypograph (HG_f) of the non-linear term of (3) and the epigraphes (EG_f) of f at $K^{max} = 2$ and the linear term, where,

$$HG_f = \{(f, s) \in \mathbb{R}^2 | f(= K s^2) \leq f(s)\} \text{ and } EG_f = \{(f, s) \in \mathbb{R}^2 | f(= K s) \geq f(s)\}.$$

The reason for this stems from the role played by the super-stable orbit at $s^* = 1/2$ and the fact that $D_0(\in \mathbb{R}^2) \geq 1$. To be more specific, consider the role of $s^* = 1/2$. That value is a member of the list of equilibria of any iterate of (3). For example, at the first bifurcation, we have:

$$f(f(s)) = K \{K s^*(1 - s^*) [1 - K s^*(1 - s^*)]\} = s^*; \tag{7}$$

at $s^* = 1/2$, $K_1 = 2$, $K_2 = 1 + (5)^{1/2}$, and $K_3 = 1 - (5)^{1/2}$ which not acceptable since $s > 0$. For the first two values, we have $s^* = 1/2$ and $s^* = 0.809$. Successive iterates are $f(f(f(s))) = s^* = 1/2$, $f(f(f(f(s)))) = s^* (= 1/2)$, etc. Thus at $s^* = 1/2$, $K^m = 2$ plays a special role that will be revealed in a moment.

In the meantime, let us first underline two important points that are necessary to characterize a market. The first refers to one of the consequence of Takens' [13] Theorem. That theorem asserts that any output (time series) of a dynamic process is sufficient to reconstruct its unknown attractor, on the one hand (see also [2], [14]. On the other, the output of a Mandelbrot- van Ness [15] process, or the fractional Brownian motion (fBm), is indexed by the Hurst' [16] exponent $H(E_x) \in (0, 1)$. However, many studies have found that $H(\cdot)$ varies with series' lengths and, more significantly, $H(\cdot)$ varies over time [17], [18], [19]. In addition, Dominique and Rivera [6] have found a close association between H and the level of investors' expectations (E_x) at certain values of K^{max} . In other words, over the range of monofractality ($3.0 \leq K^{max} \leq 3.44$), $H = a E_x^\mu$ (a is a constant greater than zero and $0 \leq \mu \leq 1$) is a smooth relation describing investors' long-term expectations. But, beyond $K^{max} > 3.44$, the relation becomes jagged, leading to jittery short-term expectations in response to the jaggedness of H . Attempting to explain this would take us far afield. It suffices to realize at this point that the mere assumption of investors' expectations becoming short-term when faced with gyrating output and uncertainty in the multi-scaling region is indeed compelling.

In the light of the ensuing discussion, we posit:

$$\begin{aligned} K^m &= K^{max} - (H/a)^{1/\mu} \tag{8} \\ &= 4 - (2 - D_0 ((H/a)^{1/\mu}) \in \mathbb{R}^2) \\ &\approx D_0 + 2. \end{aligned}$$

Hence,

$$f^m = s_{t+1} = (D_0 + 2) s_t (1 - h s_t^n). \tag{9}$$

Eq. (9) sheds light on many hereto forth nebulous concepts. Both the growth factor and variations of H are now explained. As D_0 may vary between 1 and 2, $K^m \geq 3$, falling between HG_f of f^{\max} and the EG_f 's f (of $K^m = 2$) and the linear term. Furthermore, we will show in the next section that fBm exhibits short and long-term dependences, thereby clarifying the concepts of persistence and anti-persistence. It may also be noted that with both STD and LTD , fBm cannot be a purely self-similar process because such a process cannot be stationary. Later on, we will locate fBm at the intersection of self-similar and Gaussian processes, with stationary increments and indexed by $H(E_x) \in (0, 1)$.

3.1 Firms' Efficiency

Imagine now a market characterized by $h = \eta = 1$ and $D_0 = 1$, comprising two firms, i and j , respectively. Then $K^m = 3$, $s^- = 0.500$, $s^* = 2/3$. Further, it is observed that $s_i^* = 0.400$ and $s_j^* = 0.266$. Since for the market as a whole, $h = \eta = 1$, firms' equations must satisfy the following conditions:

- a) $\eta_i \cdot \eta_j = 1$, and;
 b) $\eta_i / \eta_j = K_i / K_j$.

It is then easy to show after some manipulations that $\eta_i = 1.1034$, $\eta_j = 0.902$ satisfy the above conditions. Hence,

$$f^i = 1.6666 s_i (1 - s_i^{1.1034})$$

$$f^j = 1.3623 s_j (1 - s_j^{0.902})$$

The first indication of firm i 's superiority may be seen from the mean. That is,

$$s_i^- = [1 / (1 + 1.1034)]^{1/1.1034} = 0.5097, s_j^- = [1 / (1 + 0.902)]^{1/0.902} = 0.4902.$$

At that level, f^i reaches a maximum of 0.4038, while f^j reaches a maximum of 0.3510. But the main indication of superiority is in terms of costs per share:

$$c_i = (0.40)^{1.1034} = 0.3638, c_j = (0.266)^{0.902} = 0.3028.$$

Then, firms' efficiency is given by:

$$c_i / s_i^* = 0.9095, \quad c_j / 0.266 = 1.1383.$$

Clearly firm i is more efficient than firm j . It is then no accident that i has a higher market share in equilibrium.

3.2 The Consequences of Competition

In the above example, the equilibrium of either firm is stable, but surprisingly that of the market at $s^{m*} = 2/3 = s_i^* + s_j^*$ is only marginally stable as

scale invariance is on the verge of being broken. Suppose that either (or both) attempts to increase its own share and in so doing pushes market share upward to, say, 0.6913. At that level, $df^m / ds = |1.24|$. As a consequence, market share will bifurcate to $s^{m*} = 0.809 > 0.6913$. The excess supply will force the competitors to reduce their output until $s^{m*} = 1/2 < 0.6913$. Now excess demand will call for an increase in output. Thus, market share will tend to oscillate about 0.6913 for as long as the competitors are locked in battle for market share.

Such a situation is reminiscent of Edgeworth’s [21] bilateral monopoly problem. If both firms are satisfied at $s_i^* = 0.400$ and $s_j^* = 0.266$, they can coexist, if not, output will oscillate forever. Edgeworth concluded that the solution to his problem was unstable and indeterminate; he should have added: “ in eternal competition”.

4 Market Output

As alluded to above, attempts to evaluate experimentally the Hurst’s exponent have produced a plethora of different results ([22], [23], among others). Some authors claim long-term dependence (LTD) in financial returns ([24], [3]) while (LTD) is rejected by others ([25], [27]). Additionally, it is now known that LTD is not a property of purely self-similar processes, as self-similarity itself may have very different origins. To shed more light on the debate as to whether LTD or STD exists in financial time series some authors ([28], [29]) have proposed the more appropriate Mixed fractional Brownian motion (MfBm). Indeed, Dominique and Rivera [29] have unambiguously shown that the S&P-500 Index exhibits both LTD and STD, consistent with changes in investors’ attitude. To make this precise:

Definition 4. *The MfBm, denoted $Z_t = \sum_i b_i (X^{H_i})$, $b_i \in \mathcal{R}$, $i \in n$, $H_i \in n$, $\forall H_i \in (0,1)$, is a linear combination of quasi Gaussian processes or a superposition of n independent input streams, each with its own H .*

Using the terminology of input storage and Teletraffic, $X_t^{H_i}$ are inputs arriving as “cars” (short-term expectations) or as “trains” (long-term expectations). Also, recognizing that the market is in reality a dynamic input/output construct, it makes sense to model it as an MfBm where Z_t is observable while inputs are not; the latter are assumed to be Mandelbrot-van Ness processes. More specifically:

$$X_i^{H_i} = \{ X^H (t, \omega), t \in \mathcal{R}, \omega \in \Omega \}, \tag{10}$$

is a real-valued mixed Gaussian process, defined on a space $(\Omega, \mathcal{P}, \mathcal{F})$, indexed by $H \in (0,1)$, satisfying $E (X^{H_i} (t, \omega)) = 0, \forall t \in \mathcal{R}$, while assuming that H_i is constant over segment $i \in n$. Here E denotes the expectation with respect to the probability law \mathcal{P} for X^{H_i} , and (Ω, \mathcal{F}) is a measurable space. The probability law is somewhat

asymmetric. To show this, let first posit: ΔZ_t^+ and ΔZ_t^- as positive and negative moves at time t , respectively, while p is a probability of occurrence. Then:

1) State 1, characterized by $H > 1/2$ (or $D_0 < 1.5$): Then given,

$$\Delta Z_t^+ \rightarrow p_{11} (\Delta Z_{t+1}^+) > p_{12} (\Delta Z_{t+1}^-)$$

$$\Delta Z_t^- \rightarrow p_{12} (\Delta Z_{t+1}^-) > p_{11} (\Delta Z_{t+1}^+).$$

This means that the probability of a positive move at $t + 1$ is higher than that of a negative move if today's move is positive and $H > 1/2$, and vice versa if today's move is negative. In State 1): $p_{11} + p_{12} = 1$, but p_{11} or p_{12} may approach 1 depending on the case; something similar applies to State 2 below). Such a situation subsumes that continuity, or long-term expectations, coincides with monofractality. It may also be assumed that p_1 increases with increasing H (.). On the other hand,

2) State 2, characterized by $H < 1/2$ (or $D_0 > 1.5$):

$$\Delta Z_t^+ \rightarrow p_{22} (\Delta Z_{t+1}^-) > p_{21} (\Delta Z_{t+1}^+)$$

$$\Delta Z_t^- \rightarrow p_{21} (\Delta Z_{t+1}^+) > p_{22} (\Delta Z_{t+1}^-).$$

State 2) implies multifractality and complexity where p_2 increases with decreasing H (.) values. Hence, expectations quickly turn to short-term.

Mandelbrot and van Ness [16] distinguish between the ordinary Brownian motion and fBm at the level of covariance function. That is, for any $v > 0$ and $z > 0$, $z > v$:

$$\text{Cov} (Z_v, Z_z) = 2^{-1} [\sum_i b_i^2 (v^{2H_i} + z^{2H_i} - |v - z|^{2H_i}) > 0 (< 0) \text{ for } H_i > 1/2 (H_i < 1/2); \quad (11)$$

implying that $H > 1/2$ is associated with *persistence*, while $H < 1/2$ implies *anti-persistence*. As a consequence, monofractality (one scale) is also associated with persistence ($H > 1/2$). But we found experimentally that multifractality (multiple scales) implies both persistence and anti-persistence. Therefore, the notion to the effect that $H < 1/2$ ($H > 1/2$) means anti-persistence (persistence) needs a clarification.

The whole situation may be summarized as follows: State 1 is characterized by: $H > 1/2$ (or $D_0 < 1.5$), persistence, and $p_{11} (p_{12}) \gg p_{12} (p_{11})$. Whereas, in State 2, we have: $D_0 > 1.5$ in the complex region where $p_{22} (p_{21}) \gg p_{21} (p_{22})$, but the same probability law holds for the chaotic region which is persistent and where $D_0 < 1.5$. In other words, according to the color code used in physics, the complex region ($3.59 < K^{\max} < 3.85$) is *pink on the average*, while the interval $3.0 < K^{\max} \leq 3.44$ and the chaotic region ($3.84 < K^{\max} < 4.0$) are *black on the average*. However, *the two black colors are fundamentally different. Type 1 black is in State 1 where all orbits are stable fixed-points of periods 2^0 . Whereas Type 2 black is in State 2 where there are a few stable orbits but most are unstable and there are SPD and SDIC*. For example, at $K^{\max} = 3.84$, there is a stable orbit at s^*

= 0.958. But at $K^{\max} = 3.842$, $s^* = 0.955$ is unstable. A quasi monopolist that achieves these levels of market share is in fact very vulnerable to the slightest change in the growth parameter. Hence, Type 2 black deserves its bad reputation, for any monopole that ventures in that region may crash for reasons that would appear mysterious to its CEO.

Putting this result in the context of Figure 1, we see that the process is persistent if $D_0 < 1.5$ over the interval $3.0 \leq K^{\max} \leq 3.44$.

It is anti-persistent ($D_0 > 1.5$) over the interval $3.49 \leq K^{\max} \leq 3.84$, but persistent again over the interval $3.84 \dots < K^{\max} < 4$. Then, as the sole characteristic known as persistence is not too informative, how can the first persistent (monofractal) interval be distinguished from the second (multifractal)? We will return to this in a moment.

4.1 Testing Variations in Expectations

Figure 1 clearly shows a collapse of the H index as the process becomes self-affine, and that fact is correlated with a collapse of investors' expectations. We will attempt to support that assertion from the US capital market for which we have an exceedingly long series. That is, the Grand Microsoft Excel data set, sampled daily from January 3rd 1950 to February 28th 2011, expressed as an MfBm (Def. 4). The index was first divided into 12 segments of various lengths and according to their scaling factors. Each segment was next de-trended using logarithmic differences and filtered for white noise before computing their D_0 .

Our results are shown in Table 2 below. As it can be seen, STD is observed over 7 anti-persistent periods coinciding with economic downturns, while LTD was observed over 5 persistent periods coinciding with economic upturns. For the present purpose, however, no detailed explanation is necessary, except for two pertinent observations. That is, MfBm accounts well for the stylized facts. And the US capital market confirms the assertion to the effect that LTDs are correlated with investors' long-term expectations, while STDs are correlated with short-term expectations.

4.2 Estimating Market Conditions from Observed Data

At the outset market share was defined as $s_i = Q_i / Q$. As defined, market share does not coincide with published data, which are more properly equilibrium values, capable at best of identifying market leaders. As Q is not observable, published data do not help much in the assessment of market conditions. However, this limitation can be overcome using Figure 1 and Eq. (9). That is,

- 1) Obtain a time series or a weighted index of un-manipulated prices of the market to be analyzed;

- 2) De-trend, filter and calculate D_0 from a Wavelet multi-resolution software;
- 3) If $D_0 > 1.5$, the market falls in the anti-persistence and complex region of the map;
- 4) If $D_0 < 1.5$, the process is either a monofractal or a multifractal. To distinguish between the two, the Mandelbrot Method (see [6]) can be used to compute at least two of Renyi's ([28]) generalized dimensions. That is, the Information Dimension (D_1) and the Correlation Dimension (D_2) of Grassberger and Procaccia ([29]);
- 5) If the process or market is a monofractal, then $D_0 = D_1 = D_2$. On the other hand, if the process is located in the chaotic region, then $D_0 > D_1 > D_2$; in that case, it must be analyzed as a multifractal.

Figure 1 maps D_0 and K^{\max} . Table 1 summarizes the characteristics of f^{\max} , and Table 2 depicts STD and LTD in the S&P-500 Index from 1950 to 2011. In Figure 1: D_0 vs. K^{\max} . Over the interval $3.0 < K^{\max} < 3.44$ the process is a persistent monofractal. Over the interval $3.45 < K^{\max} < 3.84$ it is an anti-persistent multifractal. The interval $3.62 \dots \leq K^{\max} \leq 3.84 \dots$ is the Li & Yorke's period-3 window. The interval $3.84 < K^{\max} < 4.0$ is persistent- multifractal and chaotic.

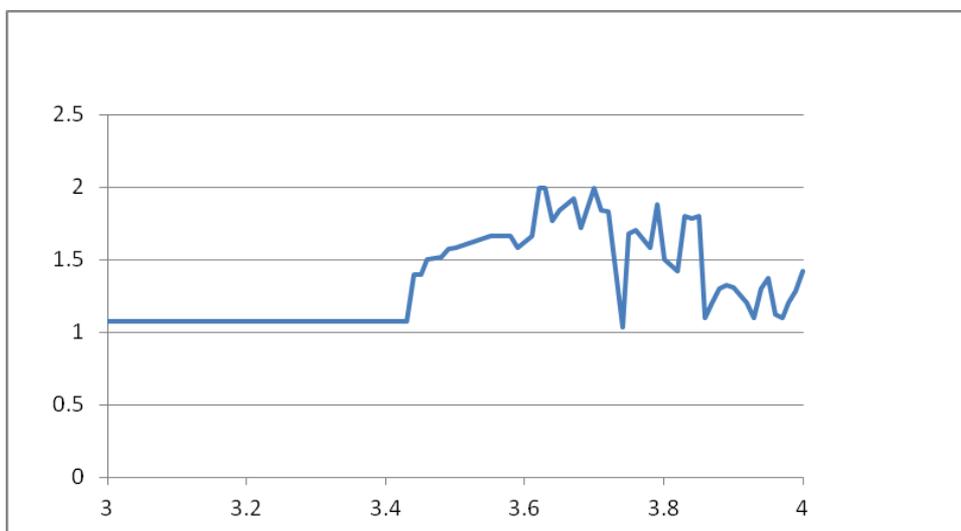


Figure 1: K^{\max}

Table 1: Theoretical and Computed Critical Values and Intervals of K^{\max} . (1) Computed with the Wavelet Multi-resolution Benoit_{TM} of Trusoft International; (2) The algorithm indicates a break with monofractality at $K^{\max} = 3.44$, but not the first bifurcation.

Table 1: Summarizes the characteristics of f^{\max}

Theoretical Value	Computed Value ⁽¹⁾	Remarks
$K < 1$	$K < 1$	$s^* = 0$
$K = 2$	$K = 2$	$s^* = 1/2$ df/ds = 0, a fundamental super-stable equilibrium
$1 < K < 3.2306$	$1 < K < 3.0$ ⁽²⁾ ...	$\forall \Gamma-2^k \in M^s, k = 0$ oh a fixed-point attractor
$K = 3.23606$	$K = 3.0...$	Second-order Return map of f^{\max} : Equilibria at $s_1^* = 0.5000, s_2^* = 0.809 ...$ Broken Symmetry at $K = 3.23$ & Beginning of Multifractality; Stretching of B at $K = 3.23$
$3.23 \dots \leq K \leq 3.45$	$3.23 \leq K \leq 3.44$	Process is a persistent monofractal, but becomes anti-persistent at $K = 3.48$
$3.23 \leq K \leq 3.5699...$	$3.48 < K \leq 3.59...$	$\Gamma-2^k \in M^s \rightarrow \Gamma-2^k \in M^u \rightarrow \Gamma-2^k \in M^s, k = 2, 3, 4, \dots$; period-doubling scenario; B becomes anti-persistent at $K = 3.48$; Small folding of B at $K^{\max} = 3.59$
$K = 3.5699$	$K = 3.59...$	Critical value for aperiodicity, $\Gamma-2^k, k \rightarrow \infty$; Attractor becomes strange; Hausdorff $\dim D_0 (\in \mathcal{R}) = 0.58; \Gamma, \Gamma^+ \rightarrow \Gamma_{np}$; that is, $\exists (\Gamma_p, \Gamma_{np}, \Gamma_d)$
$K = 3.5699$	$K = 3.60$	Second enlargement of B, starting at 3.60; Process becomes severely anti-persistent
$3.5699 < K < 3.84....$	$3.59.. < K \leq 3.85$	Intervals of complex and chaotic regimes; $M^{np} \rightarrow (M^u)$, resulting in Γ^+, Γ^- and sporadic chaos; Process is anti-persistent
$3.5699 < K < 4$	$3.59 \leq K^{\max} < 4.0$	Sensitivity to K (STP) and all values of the iterates are sensitive to noise
$3.8284 \leq K \leq 3.8414$	Period-three doubling, $\Gamma-3 \times 2^k, k = 1, 2, 3, 4, \dots$ within the Li & Yorke [12] window; few chaotic intervals
$3.8414 \leq K < 4$	$3.84 \leq K < 4.0$	Chaotic regime (Def. 3), $\exists (M^s, M^u); \Gamma^+$ are few, Γ^- are plenty; Second folding of B due to intermittency, \exists SIDC, STP, Process becomes persistent.

Table 2: LTD and STD in the S&P-500 Index from 1950-2011. The market alternates between anti-persistence and per- sistance. (1) The Hausdorff dimension in 2-D; (2) Values before crashes.

Table 2: LTD and STD in the S&P-500 Index from 1950-2011.

Period	r_i	Wavelet Points	H^i	$D_0^{(1)}$
1950-58	2126	2^{11}	0.4760 ± 0.0482	1.5240
1958-61	673	2^9	0.5890 ± 0.0410	1.4110
1961-72	2591	2^{11}	0.5220 ± 0.0321	1.4780
1972-80 ⁽²⁾	2053	2^{11}	0.2209 ± 0.0359	1.7791
1980-83	1076	2^{10}	0.2870 ± 0.0319	1.7130
1983-87 ⁽²⁾	1074	2^{10}	0.5590 ± 0.0501	1.4410
1988-92	1127	2^{10}	0.5310 ± 0.0610	1.4690
1992-97	1291	2^{10}	0.4630 ± 0.0559	1.5370
1998-02	1149	2^{10}	0.6100 ± 0.0612	1.3900
2003-07 ⁽²⁾	1024	2^{10}	0.1101 ± 0.0310	1.8899
2007-08	512	2^9	0.2811 ± 0.0326	1.7189
2009-11	535	2^9	0.1430 ± 0.0339	1.8570

5 Conclusion

This paper argues that markets are dynamic input/output constructs that are in general governed by fractal attractors. Consequently, the extant approaches to the study of the dynamics of market share's growth are unable to capture the complex nature of growth dynamics. The paper then proceeds to show instead that

the state of any market can easily be characterized in quadratic mapping. It suffices to embed an observed output (time series) of that market in the non-convex set, bounded by the hypograph of the non-linear term of the envelop map, and the epigraphes of the minimum (market) map and the linear term.

To do so, the paper develops a large class of quadratic maps that can reveal a wealth of insights. For example, at certain critical values of the growth parameter, the attracting set undergoes an enlargement precisely at the point where scale invariance is broken and multifractality ensues. At higher values of the parameter, i. e., between aperiodicity and the end of the Li and Yorke's period-3 window, the attracting set undergoes a second enlargement; that interval is characterized by complexity and sporadic chaos. Beyond the period-3 window, the attracting set abruptly shrinks; the process becomes a persistent multifractal accompanied by a greater degree of chaotic behavior.

We attribute the enlargements and folding of the attracting set to the phenomenon called intermittency arising from an incomplete intersection of the stable and unstable points. Mainly in the complex region, the system may move sufficiently far away from periodic orbits to be affected by chaos before returning to stable orbits. Some authors define a second type of intermittency as an alternation of phases of periodic orbits and unstable behavior. We found support for the first type in the complex region, and support for the second type in the period-doubling region, i. e., at points where stable orbits become unstable, or at phase shifting, i. e., at $K^m = 3.34$, $K^m = 3.51$, $K^m \approx 3.60$, etc.

These and other insights shed light on the role of the growth factor, on firms' efficiency, and on the limits of competition for market shares. Finally, the paper suggests that the application of the Mandelbrot Method can distinguish between regions of persistent-monofractality from persistent-multifractality.

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